6.728 APPLIED QUANTUM and STATISTICAL MECHANICS

TWO-Level Formula

Consider the two level Hamiltonian:

$$\bar{E} = \frac{H_{11} + H_{22}}{2} \qquad \text{and} \qquad \Delta = \frac{H_{22} - H_{11}}{2} \tag{1}$$

and $H_{12} = H_{21}^* = V$ with V real. Then the Hamiltonian with basis states u_1 and u_2 is

$$\hat{H} = \begin{pmatrix} \bar{E} - \Delta & V \\ V^* & \bar{E} + \Delta \end{pmatrix} \qquad \text{where} \qquad H_{ij} = \langle u_i | \hat{H} | u_j \rangle \tag{2}$$

with eigen energies

$$E_- = \bar{E} - \sqrt{\Delta^2 + |V|^2} \qquad \text{and} \qquad E_+ = \bar{E} + \sqrt{\Delta^2 + |V|^2} \tag{3}$$

with eigen vectors

$$\begin{pmatrix} c_1^- \\ c_2^- \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} c_1^+ \\ c_2^+ \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \tag{4}$$

and

$$\sin 2\theta = -\frac{V}{\sqrt{\Delta^2 + V^2}} \qquad \text{and} \qquad \cos 2\theta = \frac{\Delta}{\sqrt{\Delta^2 + V^2}}\,. \tag{5}$$

The eigen functions can be written in terms of the original basis set of u_1 and u_2 as

$$\psi_- = \cos\theta\, u_1 + \sin\theta\, u_2 \qquad \text{and} \qquad \psi_+ = -\sin\theta\, u_1 + \cos\theta\, u_2 \tag{6}$$

If the system is originally in state $\Psi(x, 0) = u_1(x)$, which is not an eigen state of the system, then probabilities of being in state u_2 and state u_1 are

$$p_2(t) = \frac{1}{2}\frac{V^2}{\Delta^2 + V^2} - \frac{1}{2}\frac{V^2}{\Delta^2 + V^2}\cos 2\sqrt{\Delta^2 + V^2}\,t/\hbar \tag{7}$$

and

$$p_1(t) = \frac{1}{2}\frac{2\Delta^2 + V^2}{\Delta^2 + V^2} + \frac{1}{2}\frac{V^2}{\Delta^2 + V^2}\cos 2\sqrt{\Delta^2 + V^2}\,t/\hbar \tag{8}$$