Gaussian Wavepacket supplement to 6.728 notes

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We start with a stationary (unnormalized) Gaussian wavepacket

$$e^{-x^2/2L^2} \tag{1}$$

for which the Fourier transform is

$$\sqrt{2\pi L^2}\; e^{-q^2L^2/2} \tag{2}$$

Peforming the superposition of deBroglie plane waves, we find

$$\Psi(x,t) = \sqrt{2\pi L^2}\int_{-\infty}^{\infty} e^{-\dfrac{q^2L^2}{2}}\; e^{i(qx - \dfrac{hq^2t}{2m})}\; \frac{dq}{2\pi} \tag{3}$$

This can be written

$$\Psi(x,t) = \sqrt{2\pi L^2}\int_{-\infty}^{\infty} e^{-[q^2(\dfrac{L^2}{2} + \dfrac{iht}{2m}) - iqx]}\; \frac{dq}{2\pi} \tag{4}$$

Recalling from high school algebra the trick of "completing the square", the exponent can be written

$$-\left(q\sqrt{\frac{L^2}{2} + \frac{iht}{2m}} - \frac{ix}{2\sqrt{\frac{L^2}{2} + \frac{i\hbar t}{2m}}}\right)^2 - \frac{x^2}{4\left(\dfrac{L^2}{2} + \dfrac{iht}{2m}\right)} \tag{5}$$

The last term in the exponent doesn't depend on q, so it comes outside the integral, leading to

$$\Psi(x,t) = \sqrt{2\pi L^2}e^{-\left[\dfrac{x^2}{2L^2\left(1 + \frac{i\hbar t}{mL^2}\right)}\right]}\int_{-\infty}^{\infty} e^{-\left(q\sqrt{\frac{L^2}{2}+\frac{i\hbar t}{2m}} - \frac{ix}{2\sqrt{\frac{L^2}{2}+\frac{i\hbar t}{2m}}}\right)^2}\; \frac{dq}{2\pi} \tag{6}$$

The integral is now a standard one, having the value

$$\frac{\sqrt{\pi}}{2\pi\sqrt{\frac{L^2}{2} + \frac{i\hbar t}{2m}}} \tag{7}$$

This leads to the final result In Eq. 4.18 of the notes, for the case $k = 0$.