The Thermal Domain II

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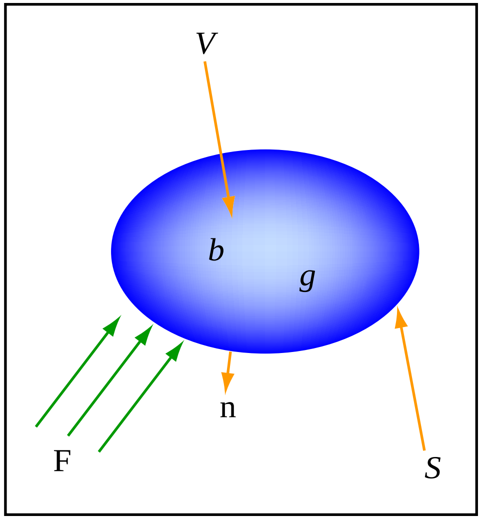
***(with thanks to SDS)**

Outline

- > Review
- > Lumped-element modeling: self-heating of resistor
- > Analyzing problems in space and 1/space
  - The DC Steady State – the Poisson equation
    - » Finite-difference methods
    - » Eigenfunction methods
  - Transient Response
    - » Finite-difference methods
    - » Eigenfunction methods
- > Thermoelectricity

The generalized heat-flow equation

- Last time we generated a general conservation equation



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$$\frac{\partial b}{\partial t} = -\nabla \cdot \mathbf{F} + G$$

- Include a flux that depends on a "force" gradient

$$\frac{d\tilde{Q}}{dt} + \nabla \cdot J_Q = \tilde{P}\Big|_{sources}$$

$$J_Q = -\kappa \nabla T$$

- And a "capacity" relation

$$\frac{\partial \tilde{Q}}{\partial T} = \tilde{C}$$

The generalized heat-flow equation

> **We get a generalized conduction equation**
- **Assume homogeneous region**

> **Applies to**
- **Heat flow**
- **Mass transport (diffusion)**
- **Squeezed-film damping**

> **Provides a rich set of solution methods**

$$\tilde{C}\frac{\partial T}{\partial t} - \kappa\nabla^2 T = \tilde{P}\Big|_{sources}$$

$$\boxed{\frac{\partial T}{\partial t} - D\nabla^2 T = \frac{1}{\tilde{C}}\tilde{P}\Big|_{sources}}$$

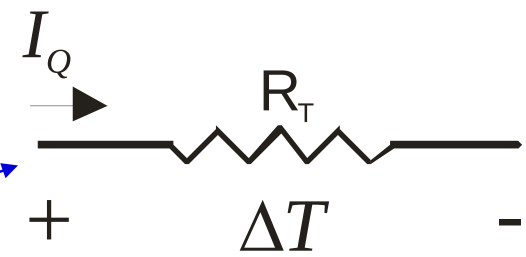
$$\boxed{[\mathrm{m^2/s}] \quad D = \frac{\kappa \ [\mathrm{W/m\text{-}K}]}{\tilde{C} \ [\mathrm{J/K\text{-}m^3}]}}$$

Thermal diffusivity

Thermal domain lumped elements

> Thermal resistor

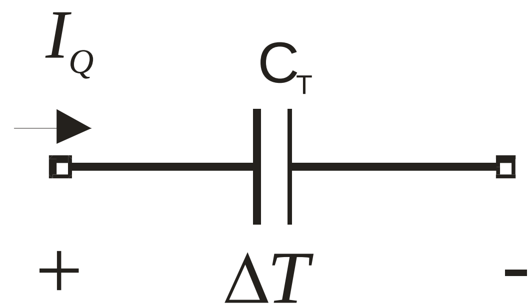
- **Resistance to heat flow**
- **Three types**
  - » **Conduction**
  - » **Convection**
  - » **Radiation**



$$R_T = \frac{1}{\kappa}\frac{L}{A}$$

> Thermal capacitor

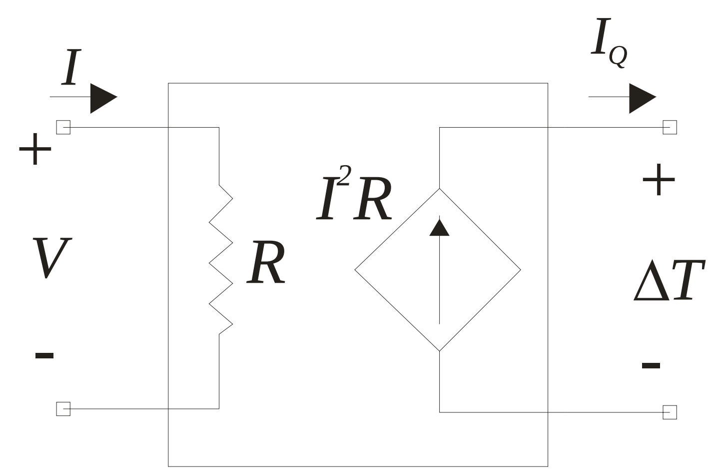
- **Store thermal energy**
- **Specific heat × volume × density**



$$C_T = \tilde{C}_m \rho_m \mathcal{V}$$

> Electrothermal transducer

- **Converts electrical dissipation into heat current**



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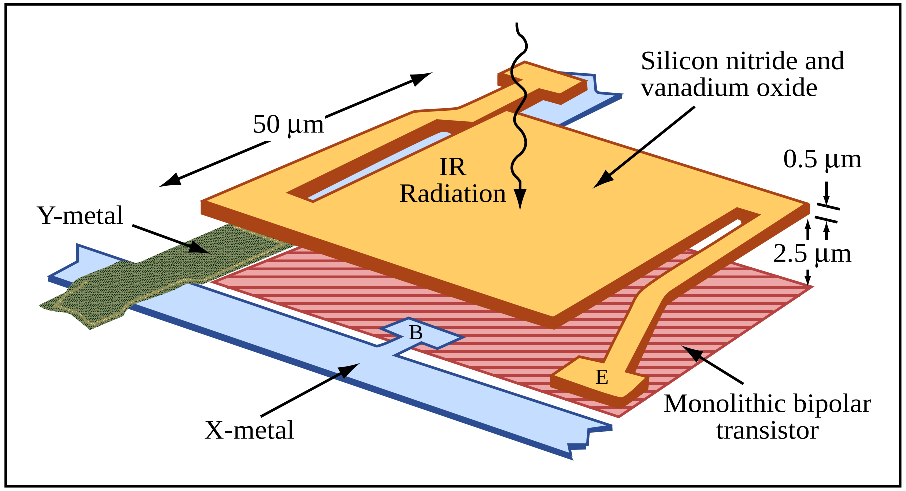
Measuring temperature with the bolometer

- So far, we know how to convert an input heat flux into a temp change
- How do we convert that temp change back into the electrical domain?



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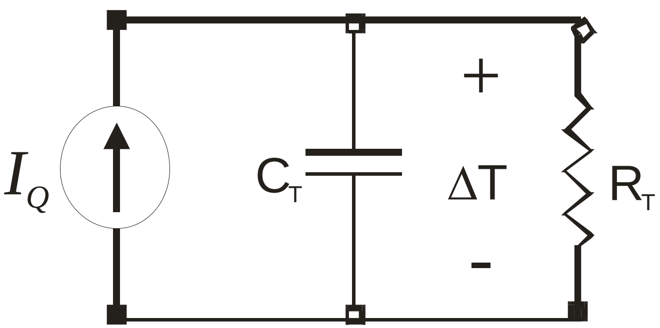


$$\Delta T = I_Q\left(R_T \,//\, \frac{1}{C_T s}\right) = \frac{R_T I_Q}{1 + R_T C_T s}$$

$$\boxed{\begin{aligned}\Delta T_{ss} &= R_T I_Q \\ \tau &= R_T C_T\end{aligned}}$$

TCR

- Resistance changes with temperature (TCR)
  - Beware, TCR is not constant!
- We can use resistor as a hotplate or a temperature sensor

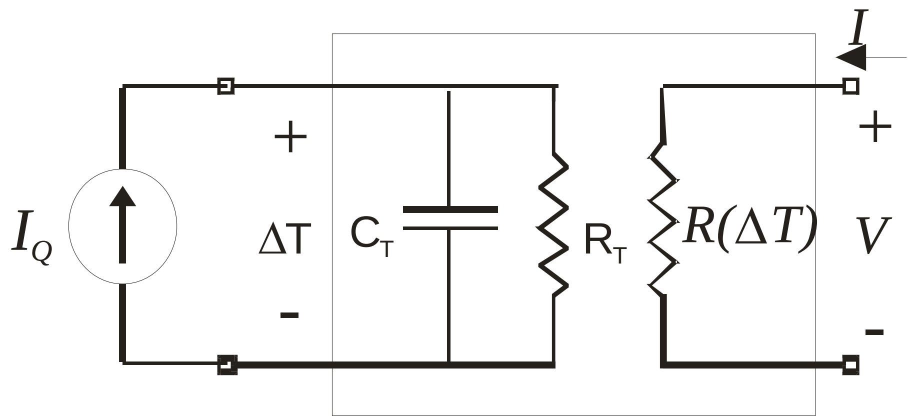
$$R(T) = R_0\left[1+\alpha_R(T-T_0)\right]$$

$$\Delta R = \frac{R-R_0}{R_0} = \alpha_R \Delta T$$

metal	resistivity [Ωcm]	TCR [10^-4/K]
Pt	10.6e-6	39.2
Nichrome	101e-6	1.7
Ni	6.84e-6	68.1
Al	2.83e-6	38
Au	2.4e-6	40
Cr	1.26e-5	30
Ti	3.84e-5	38
W	4.9e-6	45
Cu	1.72e-6	41
Fe	9.71e-6	65.1
poly-Si	10.6e-6	-12 to 12
V_2O_5		**-200**

Coupling back into the electrical domain

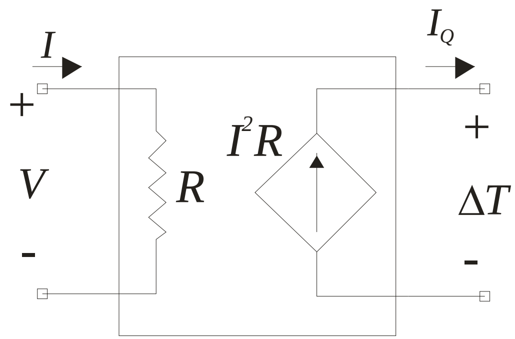
- We can define a transducer that uses TCR to convert back into electrical domain
- In order to *measure* electrical R, we need to introduce a voltage & current
- This current will couple back and induce its own ΔT

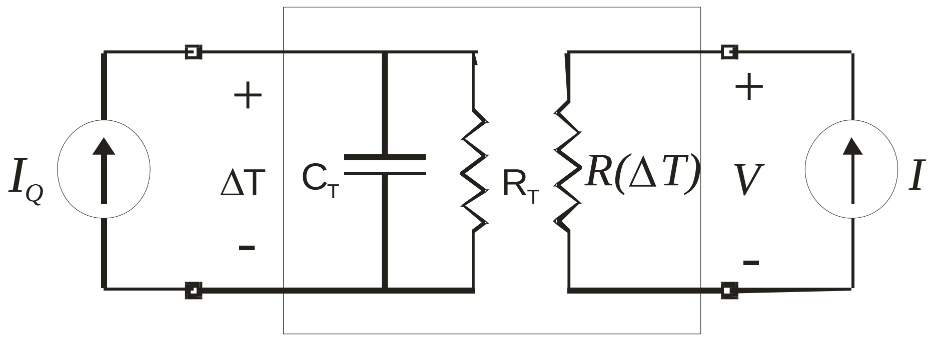


$$R(\Delta T) = R_0\left(1+\alpha_R \Delta T\right)$$

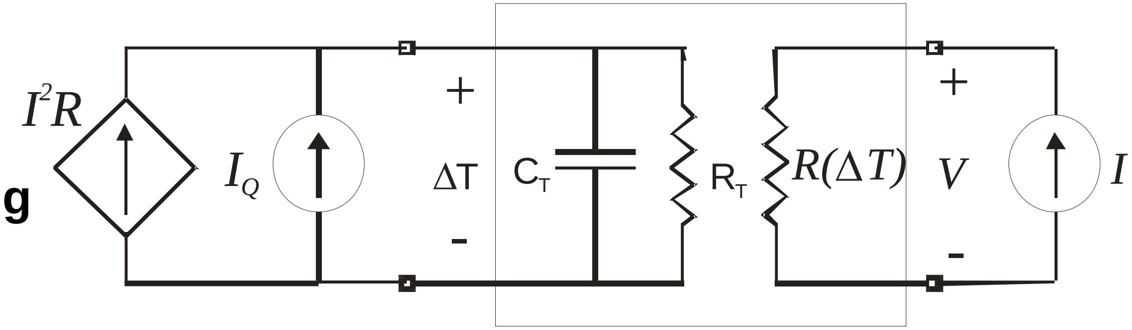
Thermo – electrical coupling

- This is our prior electrothermal transducer





- We can add in the current source due to Joule heating

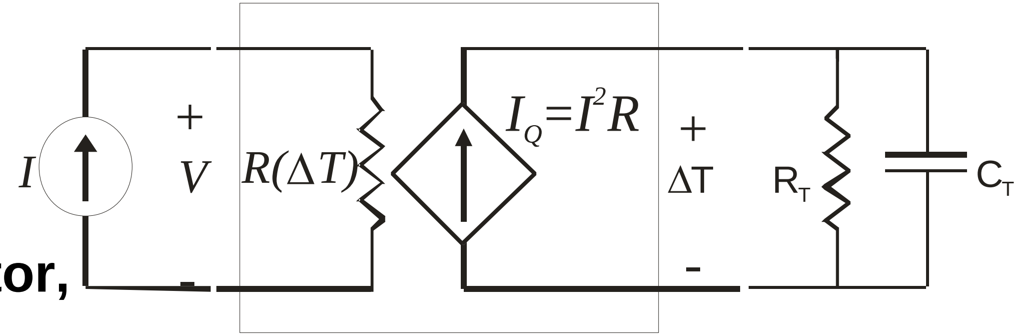


- The current source is dependent on R, which is dependent on ΔT, and so on

- What we will want is for $I_Q \gg I^2R$

Example: Self-heating of a resistor

- First, assume input $I_Q=0$
- Now, electrical port is input, temp change is output
- Two ways to drive the resistor, current source or voltage source – it sometimes matters



Current-source drive

$$\frac{\Delta T}{I_Q} = \frac{R_T}{1+R_T C_T s}$$

Expand out into D.E.

$$\Delta T\left(1+R_T C_T s\right) = I_Q R_T$$

$$\Delta T + \frac{d\Delta T}{dt} R_T C_T = I_Q R_T$$

$$I_Q = I^2 R = I^2 R_0 (1+\alpha_R \Delta T)$$

Plug in for I_Q

$$\Delta T + \frac{d\Delta T}{dt} R_T C_T = I^2 R_0 (1+\alpha_R \Delta T) R_T$$

Example: Self-heating of a resistor

> First-order system with feedback results

$$\frac{d\Delta T}{dt}R_T C_T = -\Delta T\left(1 - I^2 R_0 \alpha_R R_T\right) + I^2 R_0 R_T$$

Collect terms and rearrange

$$\frac{d\Delta T}{dt} + \Delta T\frac{\left(1 - I^2 R_0 \alpha_R R_T\right)}{R_T C_T} = \frac{I^2 R_0}{C_T}$$

Recognize D.E. form

$$\frac{dy}{dt} + ay = b$$

Pick out quantities of interest

$$\tau_I = \frac{R_T C_T}{\left(1 - I^2 R_0 \alpha_R R_T\right)}$$

$$\Delta T_{SS,I} = \frac{\frac{I^2 R_0}{C_T}}{\frac{\left(1 - I^2 R_0 \alpha_R R_T\right)}{R_T C_T}} = \frac{R_0 R_T I^2}{1 - \alpha_R R_0 R_T I^2}$$

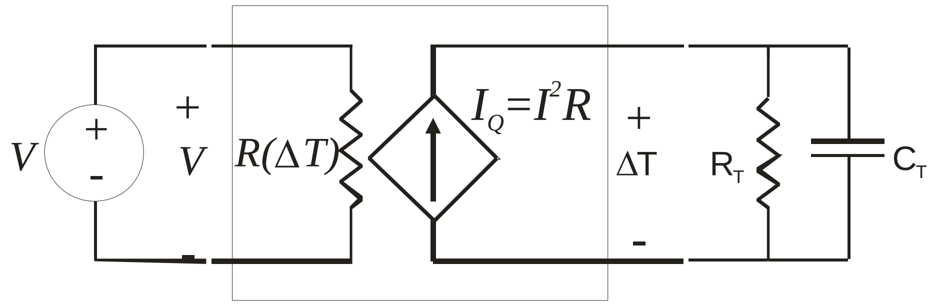
This blows up when

$$I^2 = \frac{1}{\alpha_R R_0 R_T}$$

For α_R>0

Example: Self-heating of a resistor

> **What changes for voltage-drive?**



$$\frac{\Delta T}{I_Q} = \frac{R_T}{1+R_T C_T s}$$

Same T.F.

$$\Delta T\left(1+R_T C_T s\right) = I_Q R_T$$

$$\Delta T + \frac{d\Delta T}{dt} R_T C_T = I_Q R_T$$

I_Q is now different

$$I_Q = \frac{V^2}{R} = \frac{V^2}{R_0(1+\alpha_R \Delta T)}$$

$$I_Q \approx \frac{V^2}{R_0}(1-\alpha_R \Delta T)$$

This leads to negative feedback for α_R>0

Voltage-source drive

$$\Delta T_{SS,V} = \frac{R_T V^2 / R_0}{\underline{1+\alpha_R} R_T V^2 / R_0}$$

Results of modeling

- A **positive TCR** resistor driven from a **current source** can go unstable – **fuse effect**
- When dealing with the electrostatic actuator, we observed that very different behavior was found depending on whether the system was voltage-driven or current-driven
- Here we see that, depending on the way the electrical domain couples to the thermal energy domain, it is also important to look at the drive conditions of a system.

Back to the bolometer

- Assume we want to measure I_Q=1 nW with 1% accuracy
- This limits current one can use for measurement
- For Honeywell bolometer, R_0~50 kΩ, α_R~-2%/K, R_T~10^7 K/W
- Input signal will create ΔT=10 mK
- This produces ΔR_{signal}=2x10^{-4}, or a 10 Ω resistance change
- Voltage must be < 0.7 mV

$$\Delta T_{SS,I} = \frac{R_0 R_T I^2}{1-\alpha_R R_0 R_T I^2} \approx R_0 R_T I^2$$

$$\Delta T_{SS,V} = \frac{R_T V^2 / R_0}{1+\alpha_R R_T V^2 / R_0} \approx R_T V^2 / R_0$$

$$\Delta R_{meas} = \alpha_R R_T \frac{V^2}{R_0}$$

$$\Delta R_{signal} = \alpha_R R_T I_Q$$

$$\Delta R_{meas} = \alpha_R R_T \frac{V^2}{R_0} \le 0.01 \alpha_R R_T I_Q$$

$$V \le \sqrt{\frac{I_Q R_0}{100}}$$

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Space and reciprocal space

- **We have thus far focused on “big” lumped-element modeling to design and analyze systems**
- **This isn’t the *only* way to proceed**
- **We can chop up the model into many small “lumped elements → discretize in space**
- **Or we can approximate the answer using series methods → discretize in reciprocal space**

DC Steady State

> **The Poisson Equation**

> **Boundary conditions**

- **Dirichlet – sets value on boundary**

  Fixes $T(r)\big|_{boundary}$

- **Neumann – sets slope on boundary ➔ Flux**

  Fixes $\left.\dfrac{dT}{dn}\right|_{boundary}$

- **Mixed – sets some function of value and slope**

> **The Poisson Equation is linear**

- **Can use superposition methods**

$$\frac{\partial T}{\partial t} - D\nabla^2 T = \frac{1}{\tilde{C}}\tilde{P}\Big|_{sources}$$

At steady state

$$D\nabla^2 T = -\frac{1}{\tilde{C}}\tilde{P}\Big|_{sources}$$

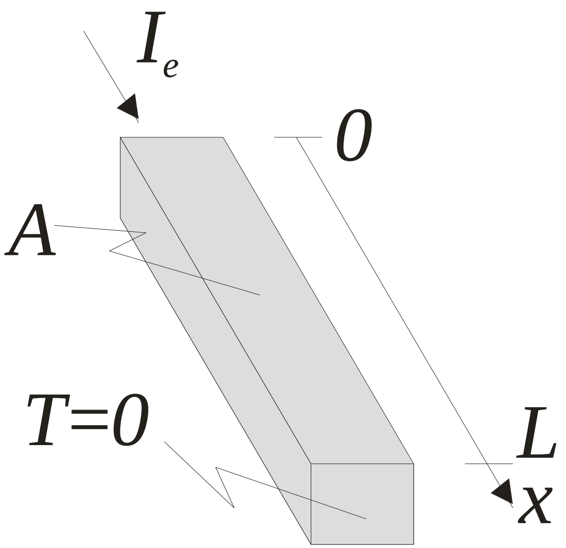
Finite-Difference Solution

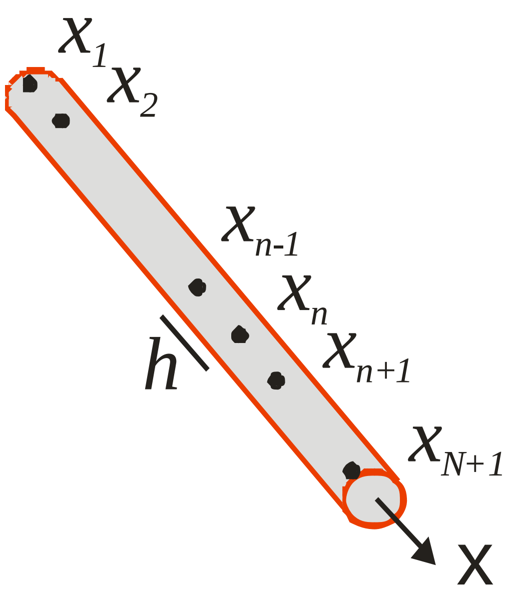
- We can generate an equivalent circuit by discretizing the equation in space
- A numerical algorithm with a circuit equivalent
- In 1-D, divide bar into N segments and N+1 nodes

$$D\nabla^2 T = -\frac{1}{\tilde{C}}\tilde{P}\Big|_{sources}$$

$$D\frac{d^2T}{dx^2} = -\frac{1}{\tilde{C}}\tilde{P}\Big|_{sources}$$

$$\frac{d^2T}{dx^2}\Big|_{x_n} \cong \frac{T(x_n+h)+T(x_n-h)-2T(x_n)}{h^2}$$





Equivalent Circuit

- Can create an equivalent circuit for this equation

$$\frac{T(x_n+h)+T(x_n-h)-2T(x_n)}{h^2}=-\frac{\tilde{P}(x_n)}{\kappa}$$

$$\left(T_{n+1}-T_n\right)+\left(T_{n-1}-T_n\right)=-h^2\frac{\tilde{P}(x_n)}{\kappa}$$

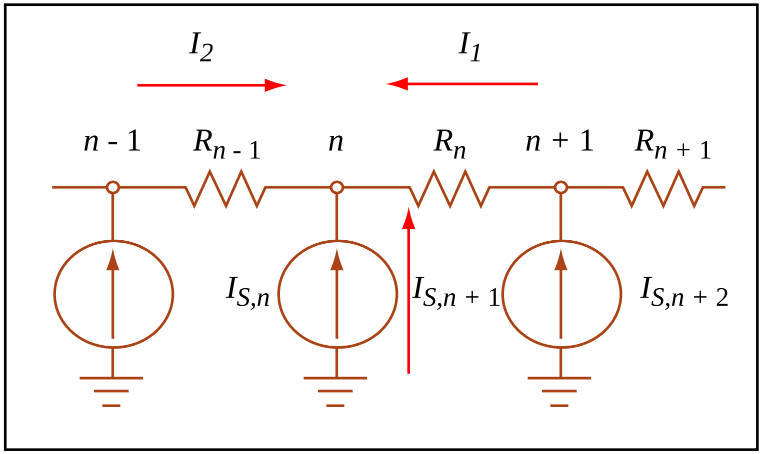
$$I_{S,n}=\left(hA\right)\tilde{P}(x_n)$$

Define local current source

$$\left(T_{n+1}-T_n\right)+\left(T_{n-1}-T_n\right)=-h\frac{I_{S,n}}{A\kappa}$$

$$\boxed{\frac{\left(T_{n+1}-T_n\right)}{R_n}+\frac{\left(T_{n-1}-T_n\right)}{R_{n-1}}+I_{S,n}=0}$$

This is KCL at a node



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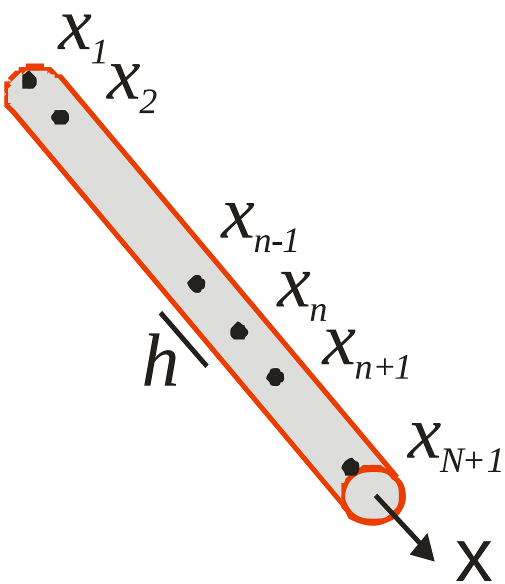
Adapted from Figure 12.1 in Senturia, Stephen D. *Microsystem Design.* Boston, MA: Kluwer Academic Publishers, 2001, p. 302. ISBN: 9780792372462.

Define local resistance

$$R_n=R_{n-1}=\frac{1}{\kappa}\frac{h}{A}$$

Equivalent Circuit

- Let's apply this to 1-D self-heated resistor



At node N:

$$\frac{\left(T_{n+1}-T_n\right)}{R_n}+\frac{\left(T_{n-1}-T_n\right)}{R_{n-1}}=-I_{S,n}$$

$G_n = 1/R_n$

$$G_n\left(T_{n+1}-T_n\right)+G_{n-1}\left(T_{n-1}-T_n\right)=-I_{S,n}$$

$G_n = G_{n+1}$

$$-GT_{n+1}+2GT_n-GT_{n-1}=I_{S,n}$$

Example: Self-heated resistor

- Set up conductance matrix
- Solve
- Very appropriate for MATLAB
- Can even generate the conductance matrix with MATLAB scripts
- At edges, impose B.C's

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & & & & & & & & 0 \\ -G & 2G & -G & \cdots & & & & & & & & 0 \\ 0 & -G & 2G & \ddots & & & & & & & & 0 \\ \vdots & \ddots & \ddots & \ddots & & & & & & & & \vdots \\ 0 & & & \ddots & 2G & -G & 0 & & & & & 0 \\ 0 & & & \ddots & -G & 2G & -G & \ddots & & & & 0 \\ 0 & & & & 0 & -G & 2G & \ddots & & & & 0 \\ \vdots & & & & & \ddots & \ddots & \ddots & & & & \vdots \\ 0 & & & & & & & \ddots & 2G & -G & 0 \\ 0 & & & & & & & \ddots & -G & 2G & -G \\ 0 & & & & & & & \cdots & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ \vdots \\ T_{n-1} \\ T_n \\ T_{n+1} \\ \vdots \\ T_{N-1} \\ T_N \\ T_{N+1} \end{pmatrix} = hA\tilde{P}_o \begin{pmatrix} 0 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{GT} = \mathbf{P}$$

Eigenfunction Solution

- **This is a standard method for solving linear partial differential equations**
- **It leads to what amount to series expansion solutions, discretized in *reciprocal* space**
- **Typically problems converge with only a few terms – THIS IS WHY IT IS USEFUL**

$$\nabla^2 T = -\frac{\tilde{P}(x)}{\kappa}$$

$$\text{Eigenfunctions of } \nabla^2: \quad \frac{d^2\psi_i}{dx^2} = \lambda_i \psi_i$$

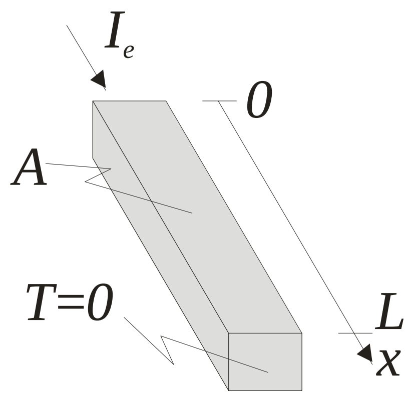
Can use any linear combination of $e^{\pm jkx}$, including $\sin(kx)$ and $\cos(kx)$

Values of k are determined by the boundary conditions

Eigenfunctions can be made orthonormal

$$\int \psi_j^* \psi_i \, dx = \delta_{ij}$$

Eigenfunction Expansion



Assume:

$$T(x)=\sum_{n=1}^{\infty}A_n\psi_n(x)$$

Eigenfunctions for this problem:

$$\psi_n(x)=c_n\sin\left(k_n x\right)$$

Apply BC at *x=0, L*:

$$\sin(k_n L)=0 \Rightarrow k_n=\frac{n\pi}{L} \quad \text{for} \quad n=1,2,3,\ldots$$

Normalize:

$$1=\int_0^L \psi_n^2(x)dx=\int_0^L c_n^2\sin^2(k_n x)dx \Rightarrow \sqrt{\frac{2}{L}}\sin(k_n x)$$

Plug into DE:

$$\frac{d^2T}{dx^2}=-\frac{\tilde{P}(x)}{\kappa}$$

$$\sum_1^{\infty}k_n^2 A_n\psi_n(x)=\frac{\tilde{P}(x)}{\kappa}$$

Eigenfunction Expansion

$$\sum_{1}^{\infty} k_n^2 A_n \psi_n(x) = \frac{\tilde{P}(x)}{\kappa}$$

Multiply by orthogonal eigenfunction and integrate:

$$\sum_{1}^{\infty} \int_0^L k_n^2 A_n \psi_n(x) \psi_m(x) dx = \int_0^L \frac{\tilde{P}(x)}{\kappa} \psi_m(x) dx$$

$$k_m^2 A_m = \frac{1}{\kappa} \int_0^L \sqrt{\frac{2}{L}} \sin\left(\frac{m\pi x}{L}\right) \tilde{P}(x) dx$$

$$A_m = \frac{1}{k_m^2 \kappa} \sqrt{\frac{2}{L}} \int_0^L \sin\left(\frac{m\pi x}{L}\right) \tilde{P}(x) dx$$

Eigenfunction Expansion

For uniform power density:

$$A_m = \frac{\tilde{P}_0}{k_m^2 \kappa} \sqrt{\frac{2}{L}} \int_0^L \sin\left(\frac{m\pi x}{L}\right) dx$$

$$A_n = \frac{\tilde{P}_0}{k_n^3 \kappa} \sqrt{\frac{2}{L}} \left(1 - (-1)^n\right)$$

$$T(x) = \sum_{n=1}^{\infty} A_n \psi_n(x)$$

$$T(x) = \sum_{n=1}^{\infty} \frac{\tilde{P}_0}{k_n^3 \kappa} \sqrt{\frac{2}{L}} \left(1 - (-1)^n\right) \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

$$T(x) = \frac{4L^2 \tilde{P}_0}{\pi^3 \kappa} \sum_{n \text{ odd}} \frac{1}{n^3} \sin\left(\frac{n\pi x}{L}\right)$$

The Details

Final answer: $$T(x) = \frac{4\tilde{P}_o L^2}{\kappa \pi^3} \sum_{n \text{ odd}} \frac{\sin\left(n\pi x / L\right)}{n^3}$$

Power density: $$\tilde{P}_o = \frac{I_e^2 R}{volume} = \frac{I_e^2}{\sigma_e A^2}$$

At *x*=*L/2*: $$T_{\max} = \left(\frac{4}{\pi^3}\right) \frac{I_e^2 L^2}{\sigma_e \kappa A^2} \left[1 - \frac{1}{3^3} + \frac{1}{5^3} + \cdots \right]$$

Even if we consider only the first term in the expansion, we find

$$T_{\max} = \left(\frac{1}{7.75}\right) \frac{I_e^2 L^2}{\sigma_e \kappa A^2}$$ compared to the exact solution of $$\left(\frac{1}{8}\right) \frac{I_e^2 L^2}{\sigma_e \kappa A^2}$$

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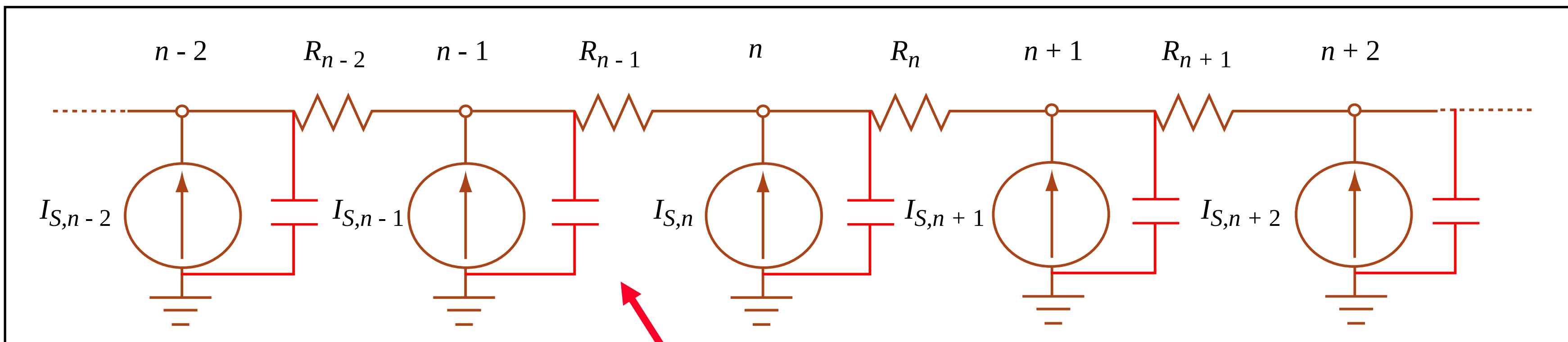
Transient Modeling

> Finite-difference method

- **Simply add a thermal capacitance to ground at each node of the finite-difference network. These circuits can be analyzed with SPICE or other circuit simulators.**



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Adapted from Figure 12.1 in Senturia, Stephen D. *Microsystem Design.* Boston, MA: Kluwer Academic Publishers, 2001, p. 302. ISBN: 9780792372462.

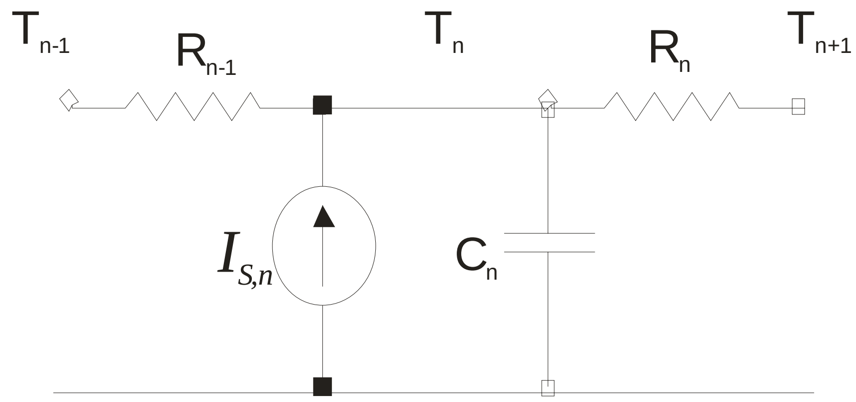
$$C = hA\rho_m \tilde{C}_m$$

Node volume

Transient Modeling

> Finite-difference method

- **What does the matrix representation look like now?**



$$I_{S,n} + G_{n-1}\left(T_{n-1} - T_n\right) + G_n\left(T_{n+1} - T_n\right) - C_n \dot{T}_n = 0$$

$$-G_{n-1}\left(T_{n-1} - T_n\right) - G_n\left(T_{n+1} - T_n\right) + C_n \dot{T}_n = I_{S,n}$$

$$\Downarrow$$

$$\mathbf{GT} + \mathbf{C}\dot{\mathbf{T}} = \mathbf{P}$$

$$\dot{\mathbf{T}} = \mathbf{AT} + \mathbf{BP}$$

Transient Modeling (continued)

> **Eigenfunction method:**

- **Spatial response same as before**
- **Use impulse response in time to eventually get Laplace transfer function**
- **Use separation of variables to separate space and time**

$$T(x,t) = \hat{T}(x)Y(t)$$

$$\frac{dT}{dt} - D\nabla^2 T = \frac{1}{\tilde{C}}\tilde{P}\Big|_{sources}$$

$$\tilde{P}(x,t) = \tilde{Q}_0(x)\delta(t) \qquad \text{[J/m}^3\text{]}$$

Separate variables

$$\frac{dT}{dt} - D\nabla^2 T = 0 \qquad t > 0^+$$

$$\hat{T}(x)\frac{dY(t)}{dt} - DY(t)\frac{d^2\hat{T}(x)}{dx^2} = 0$$

$$D\frac{1}{\hat{T}(x)}\frac{d^2\hat{T}(x)}{dx^2} = -\alpha = \frac{1}{Y(t)}\frac{dY(t)}{dt}$$

Transient Modeling (continued)

> Eigenfunction method:

- **Time response is a sum of decaying exponentials**
- **Time and space are linked via eigenvalues**

$$D\frac{1}{\hat{T}(x)}\frac{d^2\hat{T}(x)}{dx^2} = -\alpha = \frac{1}{Y(t)}\frac{dY(t)}{dt}$$

$$Y(t) = e^{-\alpha t}$$

$$T(x,t) = \hat{T}(x)e^{-\alpha t}$$

$$D\frac{d^2\hat{T}}{dx^2} = -\alpha\hat{T}$$

$$\hat{T}(x) = \sum_n A_n\sqrt{\frac{2}{L}}\sin(k_n x)$$

$$k_n^2 D = \alpha \Rightarrow k_n^2 = \frac{\alpha}{D} = \left(\frac{n\pi}{L}\right)^2$$

Transient Modeling (continued)

> Eigenfunction method:

- **Match I.C. at t=0 to get series coefficients**
- **T(x,0) is related to instantaneous heat input and heat capacity**

$$T(x,t) = \sum_n A_n \sqrt{\frac{2}{L}} \sin(k_n x) e^{-\alpha_n t}$$

$$T(x,0) = \sum_n A_n \sqrt{\frac{2}{L}} \sin(k_n x) = \frac{\tilde{Q}_0}{\tilde{C}}$$

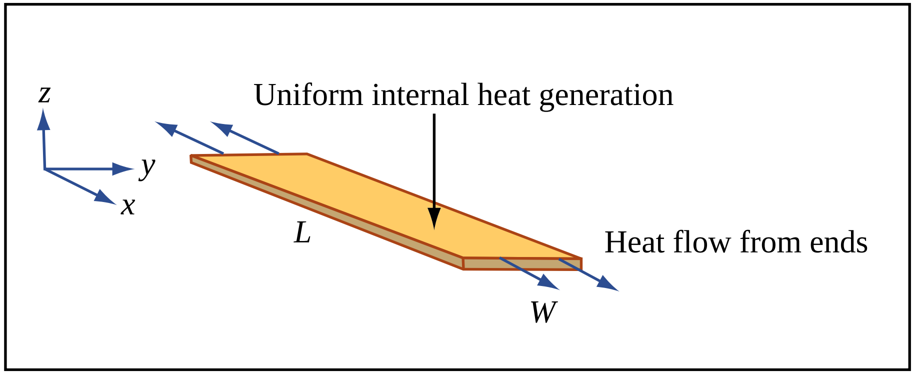
$$A_n = \frac{\tilde{Q}_0}{\tilde{C}} \sqrt{\frac{2}{L}} \int_0^L \sin(k_n x) dx$$

$$A_{n,odd} = \frac{\tilde{Q}_0}{\tilde{C}} \sqrt{\frac{2}{L}} \frac{2}{k_n} = \frac{\tilde{Q}_0}{\tilde{C}} \sqrt{\frac{2}{L}} \frac{2L}{n\pi}$$

$$\boxed{\hat{T}(x,t) = \sum_{n,\,\mathrm{odd}} \frac{4}{n\pi} \frac{\tilde{Q}_0}{\tilde{C}} \sin(k_n x) e^{-\alpha_n t}}$$

Example: Impulse Response

- Uniformly heated bar, an impulse in time
- Result is a series of decaying exponentials in time



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Adapted from Figure 12.3 in Senturia, Stephen D. *Microsystem Design.* Boston, MA: Kluwer Academic Publishers, 2001, p. 308. ISBN: 9780792372462.

$$T(x,t) = \frac{\tilde{Q}_0}{\tilde{C}} \sum_{n\,\mathrm{odd}} \left(\frac{4}{\mathrm{n}\pi} \right) \sin\left(\frac{n\pi x}{L} \right) e^{-\alpha_n t}$$

$$\text{where} \quad \alpha_{\mathrm{n}} = \frac{n^2 \pi^2 D}{L^2}$$

lower spatial frequencies decay slower

Using the Eigenfunction Solution

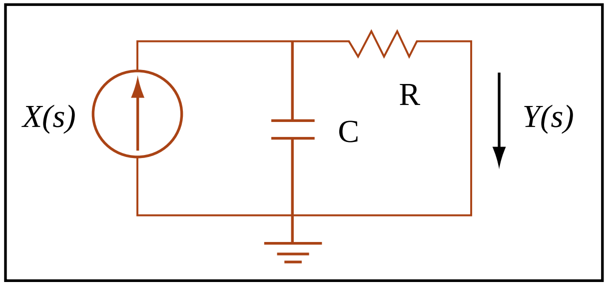
- We can go from solution to equivalent circuit
- First, we will lump
  - Heat current conducted out as output
  - Choose heat current source as input $\tilde{P} = \tilde{Q}_0 \delta(t)$
- Then take Laplace
- Then identify equivalent circuit for 1st order system
  - This is NOT unique

$$I_Q = -\kappa \left(\int_0^W \int_0^H \left.\frac{\partial T}{\partial x}\right|_{x=0} dzdy - \int_0^W \int_0^H \left.\frac{\partial T}{\partial x}\right|_{x=L} dzdy \right)$$

$$I_Q(t) = \left(\kappa WH \frac{8}{L} \sum_{n\ \mathrm{odd}} e^{-\alpha_n t} \right) \frac{\tilde{Q}_0}{\tilde{C}}$$

$$I_{Q,n}(s) = \frac{8\kappa WH}{\alpha_n L} \frac{1}{1 + s/\alpha_n} \frac{\tilde{Q}_0}{\tilde{C}}$$

$$Y(s) = \frac{1}{1 + s/\alpha_n} X(s)$$



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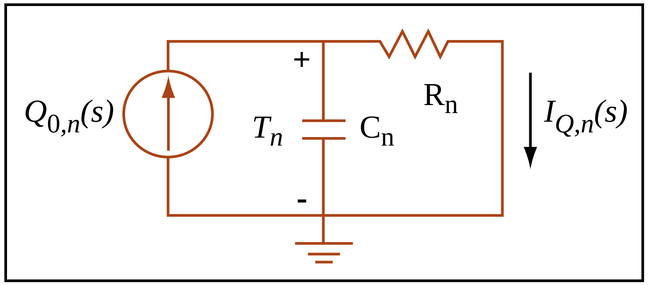
Adapted from Figure 12.4 in Senturia, Stephen D. *Microsystem Design*. Boston, MA: Kluwer Academic Publishers, 2001, p. 310. ISBN: 9780792372462.

Using the Eigenfunction Solution

- ***Each term in the eigenfunction solution has a simple circuit representation***
- **This means that if the eigenfunction solution converges with a few terms, the lumped circuit is very simple**



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Adapted from Figure 12.4 in Senturia, Stephen D. *Microsystem Design*. Boston, MA: Kluwer Academic Publishers, 2001, p. 310. ISBN: 9780792372462.

$$C_n = (\text{mode shape})(\text{volume})\tilde{C}$$

$$C_n = \tilde{C}\int_0^H\int_0^W\int_0^L \sin\left(\frac{n\pi x}{L}\right)dxdydz = \boxed{\frac{2}{n\pi}LHW\tilde{C}}$$

$$\frac{1}{R_nC_n} = \alpha_n = \frac{n^2\pi^2 D}{L^2}$$

$$\Downarrow$$

$$\boxed{R_n = \left(\frac{1}{n\pi}\right)\frac{L/2}{\kappa WH}}$$

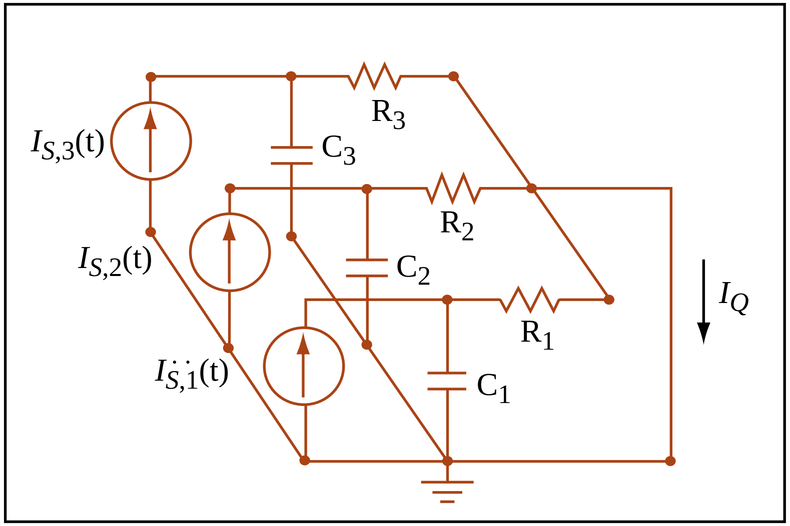
$$Q_{0,n}(s) = \frac{8\kappa WH}{\alpha_n L}\frac{\tilde{Q}_0}{\tilde{C}} = \frac{8\kappa WHL}{n^2\pi^2 D}\frac{\tilde{Q}_0}{\tilde{C}}$$

$$\boxed{Q_{0,n}(s) = \left(\frac{8}{n^2\pi^2}\right)(WHL)\tilde{Q}_0}$$

$$Q_{0,n}(t) = \left(\frac{8}{n^2\pi^2}\right)(WHL)\tilde{Q}_0\delta(t)$$

A Three-Mode Equivalent Circuit

> **For the first three terms in the eigenfunction expansion, we combine the three single-term circuits appropriately**



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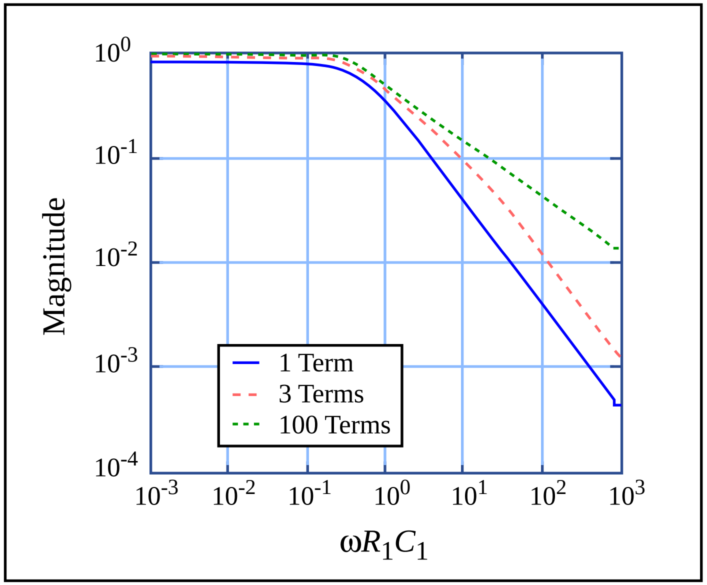

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Adapted from Figure 12.6 in Senturia, Stephen D. *Microsystem Design.* Boston, MA: Kluwer Academic Publishers, 2001, p. 313. ISBN: 9780792372462.

Outline

> **Review**

> **Lumped-element modeling: self-heating of resistor**

> **The DC Steady State – the Poisson equation**

- **Finite-difference methods**
- **Eigenfunction methods**

> **Transient Response**

- **Finite-difference methods**
- **Eigenfunction methods**

> **Thermoelectricity**

Microscale temperature measurement/control

> We have seen that a resistor can be used as a temperature sensor and hotplate

> There are other techniques to measure or control temperature at microscale
  - Couple temperature to material properties

> Sensors
  - TCR: temperature → resistance change
  - Thermal bimorph: temperature → deflection
  - Thermoelectrics: temperature → induced voltage

Coupled Flows

- In an ideal world, one driving force creates one flux

$$J_Q = -\kappa \nabla T$$

- In *our* world, multiple forces create multiple fluxes

$$J_e = -\sigma_e \nabla \varphi$$

  - Drift-diffusion in semiconductors or electrolytes

$$J_e = -z_n q_e D_n \nabla n - q_e n \mu_n \nabla \varphi$$

- In general, all the different fluxes are coupled

$$J_i = \sum_{j=1}^{n} L_{ij} F_j$$

- If you set it up right, the L_{ij} matrix is reciprocal
  - The Onsager Relations

Quantities in the Onsager Relations

- To explain thermoelectrics, we must look at coupling between heat flow and electric field
- This is written in a standard form

$$J_e = -L_{11}\nabla\varphi - L_{12}\nabla T$$

$$\frac{J_Q}{T} = -L_{21}\nabla\varphi - L_{22}\nabla T$$

Resistivity

Seebeck coefficient

$$-\nabla\varphi = \rho_e J_e + \alpha_S \nabla T$$

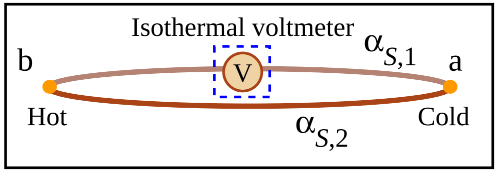
$$J_Q = \Pi J_e - \kappa\nabla T$$

Peltier coefficient

Thermal conductivity

Thermocouples

- **Analyze the potential gradient around a closed loop under the assumption of zero current (J_e=0)**
- **Thermocouple voltage depends on the difference in Seebeck Coefficient between the two materials, integrated from one temperature to the other**
- **It is a BULK EFFECT, not a junction effect**
- **It is possible to make thermocouples by accident when using different materials in MEMS devices in regions that might have temperature gradients!**



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$$-\nabla\varphi = \alpha_S \nabla T$$

$$\Downarrow$$

$$V_{ab} = \int_{T_a}^{T_b} \alpha_S(T)\,dT$$

Go around the loop

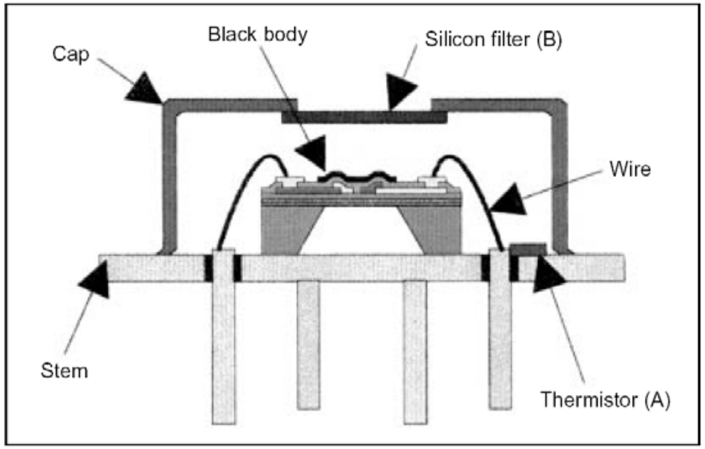
$$V_{TC} = \int_{T_C}^{T_H} \left(\alpha_{S,2} - \alpha_{S,1}\right) dT$$

$$\boxed{V_{TC} = \left(\alpha_{S,2} - \alpha_{S,1}\right)\Delta T}$$

For small temp rises

MEMS Thermocouples

- Many thermocouples in series create higher sensitivity (V/K)
- These are known as thermopiles
- In MEMS thermopiles, often use Al/Si or Al/polySi
- Able to get good thermal isolation of sensing element
- Number of thermocouples is limited by leg width
  - Increasing leg width decreases thermal resistance and thus temperature response



Courtesy of Thermometrics Corporation. Used with permission.

Thermometrics commercial Si thermopile

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Conclusions

> **The thermal domain is a great way to transfer energy around**
  - **Except that you have to pay the tax**

> **We can model thermal problems using**
  - **Equivalent circuits via lumped element models *in space***
    - **"Big" and "small"**
  - **Equivalent circuits via lumped element models *in reciprocal space***

For Further Information

> **Introduction to Heat Transfer, Incropera and DeWitt**

> **Analysis of Transport Phenomena, William Deen**

> **Solid-State Physics, Ashcroft and Mermim**