| 6.890 | Lecture 11 | Oct. 9, 2014 |
|---|---|---|

## Recall from L10:

**L-reduction:**

$$A \rightarrow B$$
$$x \xrightarrow{f} x' = f(x)$$
$$\vdots$$
$$g(x, y') = x' \xleftarrow{g} y'$$

① $OPT_B(x') = O(OPT_A(x))$ ← $\le \alpha \cdot$

② $|cost_A(y) - OPT_A(x)| = O(|cost_B(y') - OPT_B(x')|)$ ← $\le \beta \cdot$

[Papadimitriou & Yannakakis – JCSS 1991]

⇒ PTAS-reduction
- for minimization: $\delta(\varepsilon) = \varepsilon / \alpha\beta$ (AP-reduction)

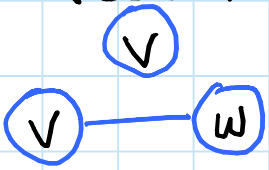
## APX-complete problems so far:

- Max E3SAT-E5
- Max 3SAT-3
- Independent set
- Vertex cover
- Dominating set

} bounded degree (Independent set, Vertex cover, Dominating set)

## Max 2SAT: [Papadimitriou & Yannakakis – JCSS 1991]

- L-reduction from Independent set, bounded deg.




⇨ $v$ — ∀ vertex $v$

⇨ $\neg v \vee \neg w$ — ∀ edge $\{v,w\}$

- never worth violating edge constraint: could violate either vertex at same cost
  - ⇒ solution gives an indep. set
  - ⇒ $OPT_{2SAT} = \underbrace{OPT_{IS}}_{\Theta(|V|)} + \underbrace{\#\text{edges}}_{\Theta(|V|)}$ — bounded degree

## Max E2SAT-E3 → [Berman & Karpinski – ICALP 1999]

## Max NAE 3SAT: [Papadimitriou & Yannakakis – JCSS 1991]

- strict-reduction from Max 2SAT

  $x \vee y$ ⇨ $NAE(x, y, a)$ ↰ same in all clauses

- by flipping, can assume $a = \emptyset$
- score = # $(x,y)$s where $x$ or $y = 1$

## Max cut:

[Papadimitriou & Yannakakis – JCSS 1991]

= max positive 1-in-2SAT

= max positive XOR-SAT

- L-reduction from Max NAE 3SAT:
  - clause gadget: 2 points if satisfied, 0 else
  - variable gadget: never hurts to put $x_i$ & $\bar{x}_i$ in opposite sides

$\Rightarrow OPT_{cut} = 2 \cdot (\sum_i \#\text{occurrences of } x_i$ → ≤ 3·#clauses

$+ \#\text{satisfied clauses})$

$= \Theta(OPT_{NAE})$ → ≥ ½ #clauses

- degree-3 possible

[Berman & Karpinski – ICALP 1999]

≤ max E2-LIN-$\mathbb{Z}_2$-3

E2 ↓ = 2 literals/eqn.

LIN-$\mathbb{Z}_2$: linear eqns. over $\mathbb{Z}_2$

-3: ↳ 3 eqns./variable

# Max/min CSP / Ones:  [Khanna, Sudan, Trevisan, Williamson — SICOMP 2001]

(CSP → # clauses; Ones → # true variables)

- analog to Schaefer Dichotomy
- given allowable clause functions
- instance can be weighted or not
- e.g.: Max E2SAT = Max CSP($x_1 \vee x_2$, $\bar{x}_1 \vee x_2$, $x_1 \vee \bar{x}_2$, $\bar{x}_1 \vee \bar{x}_2$)
  Max Cut = Max CSP($x_1$ XOR $x_2$)
  Max Clique = Max Ones($x_1$ NAND $x_2$)
- Max CSP
  - ∈ PO if setting all vars. false or all vars. true satisfies all clause types
  - ∈ PO if all clauses in DNF have 2 terms, one all positive & one all negative
  - APX-complete otherwise
- Max Ones:
  - ∈ PO if setting all vars. true satisfies all
  - ∈ PO if CNF of Dual-Horn subclauses (≤1 negated)
  - ∈ PO if ≤2-X(N)OR-SAT: linear eqns., 2 terms, over $\mathbb{Z}_2$
  - APX-complete if ≤X(N)OR-SAT (not 2-)
  - Poly-APX-complete if CNF of Horn subclauses
  - Poly-APX-complete if 2CNF
  - Poly-APX-complete if setting all or all but one variable false satisfies each constraint
  - 0 vs. >0 NP-hard if setting all vars. false satisfies (& not previous case)
  - feasibility NP-hard if none of above

- Min CSP:
  - ∈PO if setting all vars. false or all vars. true satisfies all clause types
  - ∈PO if all clauses in DNF have 2 terms, one all positive & one all negative
  - APX-complete if OR(O(1) variables), $\neg x_1 \vee x_2$
    - (OR(O(1) variables): O(1)-hitting set; $\neg x_1 \vee x_2$: implication)
  - Min UnCut-complete if ≤2-X(N)OR-SAT
    - Min UnCut: Min CSP(XOR) — APX-hard & O(log n)-approx.
  - Min 2CNF-Deletion-complete if 2CNF
    - Min 2CNF-Deletion: Min CSP(OR, NAND) ~ APX-hard & O(log n loglog n)-apx.
  - Nearest Codeword-complete if ≤X(N)OR-SAT (not 2-)
    - Nearest Codeword: Min CSP($x_1 \oplus x_2 \oplus x_3$, $\overline{x_1 \oplus x_2 \oplus x_3}$) — $\Omega(2^{\log^{1-\varepsilon} n})$-inapprox.
  - Min Horn Deletion-complete if Horn or Dual-Horn
    - Min Horn Deletion: Min CSP($\bar{x}_1 \vee x_2 \vee x_3$) — $\Omega(2^{\log^{1-\varepsilon} n})$-inapprox., ∈Poly-APX
  - 0 vs. >0 is NP-complete otherwise
- Min Ones:
  - ∈PO if setting all vars. false satisfies all
  - ∈PO if CNF of Horn subclauses (≤1 positive)
  - ∈PO if ≤2-X(N)OR-SAT
  - APX-complete if 2CNF
  - APX-complete if O(1) hitting set + implication
  - Nearest Codeword-complete if ≤X(N)OR-SAT (not 2-)
  - Min Horn Deletion-complete if CNF of Dual-Horn
  - Poly-APX-complete if all vars. true satisfies — if weighted: hard to approximate by any factor
  - feasibility NP-hard otherwise

# Another APX-completeness series:

## Max. independent set in 3-regular 3-edge-colorable graphs

[Chlebík & Chlebíková — CIAC 2003]

## Max. 3DM-E2:

- given triples ⊆ $A \times B \times C$ (disjoint)
- solution = subset of triples not repeating any item $\in A \,\dot\cup\, B \,\dot\cup\, C$
- each item appears ≤ twice
- strict-reduction from previous problem:
  - edge color classes → A, B, C
  - vertex → triple

## Max. edge matching puzzles:

[Antoniadis & Lingas — SOFSEM 2010]

- goal: maximize # matching edges
- ∈ APX (max. matching gives ≥ n/8 matches)
- L-reduction from previous problem
  - 2×n

$\Theta(1)$-approximable ($\in$ APX \ PTAS) but not APX-complete: [Crescenzi, Kann, Silvestri, Trevisan - SICOMP 1999]
(unless polynomial hierarchy collapses)
PH

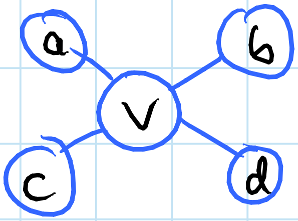
Bin packing: given n numbers & bin size B, min. # bins to store the numbers
- has asymptotic PTAS (+1 additive error)

Min. max.-degree spanning tree

Min. edge coloring

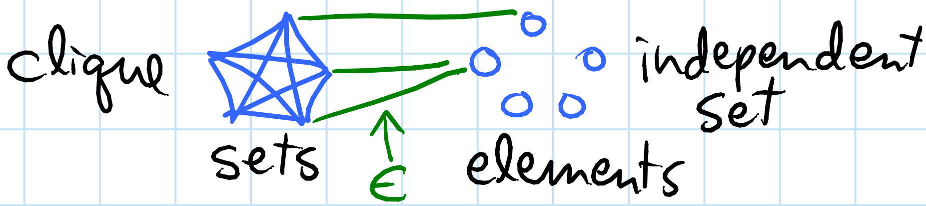
**Log-APX-complete:** (A-reductions: $y'$ c-approx. $\Rightarrow$ $y$ $O(c)$-approx.)

- set cover
- dominating set [Escoffier & Paschos — TCS 2006]
- strict-reduction from dom. set to set cover:




$\Rightarrow$ $S_v = \{v, a, b, c, d\}$

- strict-reduction from set cover to dom. set:




  - never need to choose element: take a set ⇒

**Token reconfiguration:** [Calinescu, Dumitrescu, Pach — LATIN 2006]

- given initial & goal token placements
- move = slide pebble along empty path
- goal: min. # moves
- APX-hard for unlabeled & labeled tokens
  - L-reductions from Set Cover
- 3-approx. for unlabeled
- motivation: $15 = n^2 - 1$ puzzle
  - NP-hard & ∈ APX [Ratner & Warmuth 1990]

Poly-APX-complete: max. independent set
& max clique (complement)

(PTAS-reductions) [Bazgan, Escoffier, Paschos – TCS 2005]

Exp-APX-complete: nonmetric Traveling salesman
↳ $2^{n^{O(1)}}$ [Escoffier & Paschos – TCS 2006]

## NPO-complete: THE HARDEST! (AP-reductions)

[Crescenzi, Kann, Silvestri, Trevisan – SICOMP 1999]

### Max./min. weighted SAT (AKA "ones")

- given CNF formula
- given nonneg. weight $w_i$ of each var. $x_i$
- solution = satisfying assignment (NP-hard!)
- cost = $\sum_i w_i x_i$ (can max with 1)

### Max/min 0-1 linear programming:

- given integer matrix A, vectors b & c
- max/min $c \cdot x$ → 0-1 vector

  subject to $Ax \geq b$

## NPO PB-complete: above with integer → {0, 1} (or poly. bounded integer)

- $n^{1-\varepsilon}$-inapproximable even with trivial solutions [Jonsson – IPL 1998]
- min. independent dominating set ← [Kann – NJC 1994]
- shortest computation in nondet. Turing machine ← [Kann – NJC 1994]
- longest induced path ← [Berman & Schnitger – I&C 1992]
- longest path with forbidden pairs ← [Berman & Schnitger – I&C 1992]

6.890 Algorithmic Lower Bounds: Fun with Hardness Proofs
Fall 2014