Random Networks and Percolation

- Percolation, cascades, pandemics
- Properties, Metrics of Random Networks
- Basic Theory of Random Networks and Cascades
- Watts Cascades
- Analytic Model of Watts Cascades

Types of Percolation Models

- "Short loop" models (Stauffer, Grimmett, Morris) usually assume regular network structure or same nodal degree for all nodes
- "Long loop/no loop" models (Newman, Calloway, Watts) usually assume random tree-like structure
- "Collective action" models (Schelling, Granovetter) assume $k = n$ (all nodes see all other nodes)
- "Local action" models assume $z << n$ (Watts, etc.)
- "Threshold models" (Morris, Schelling, Granovetter, Watts) assume that a node changes state when more than a threshold **fraction** of neighboring nodes have changed state
  - Threshold models are equivalent to deterministic two-person games (Lopez-Pintado) (Morris)
  - Disease spreading (SIR) assumes a threshold **number** of neighbors
- Mixed collective/local models have also been proposed (Valente) for diffusion of innovations

Percolation Contexts

- Spread of diseases (Watts and others)(local)
- Propagation of rumors (Newman, Watts, Calloway)(local)-scary talk about surprises
- Success of “blockbusting” (Schelling)(collective)
- Decision to join a riot (Granovetter)(collective)
- Adoption of innovations (Rogers, Valente)(both)
- In each case, nodes are assumed to be different in their susceptibility - an important issue for sociologists
- **Thresholds** are used to model these differences

Diffusion of Pandemic Diseases

- Model assumes disease starts from a point and travels in two modes: local commuting and international air travel
- Disease follows SIR or SIS model, with parameters that need to be estimated for each outbreak
- Procedure is to run the model with different trial parameters and see which ones best match time of outbreaks in different main airline destinations
- Model has been built up over about 10 years of IATA, census, and local transportation data
- Prediction that H1N1 would peak in US in October at low levels
- Prediction that it originated in Mexico, not a pig farm in MN, etc.
- http://cnets.indiana.edu/tag/epidemic-modeling

Diffusion of the Black Death: Slow

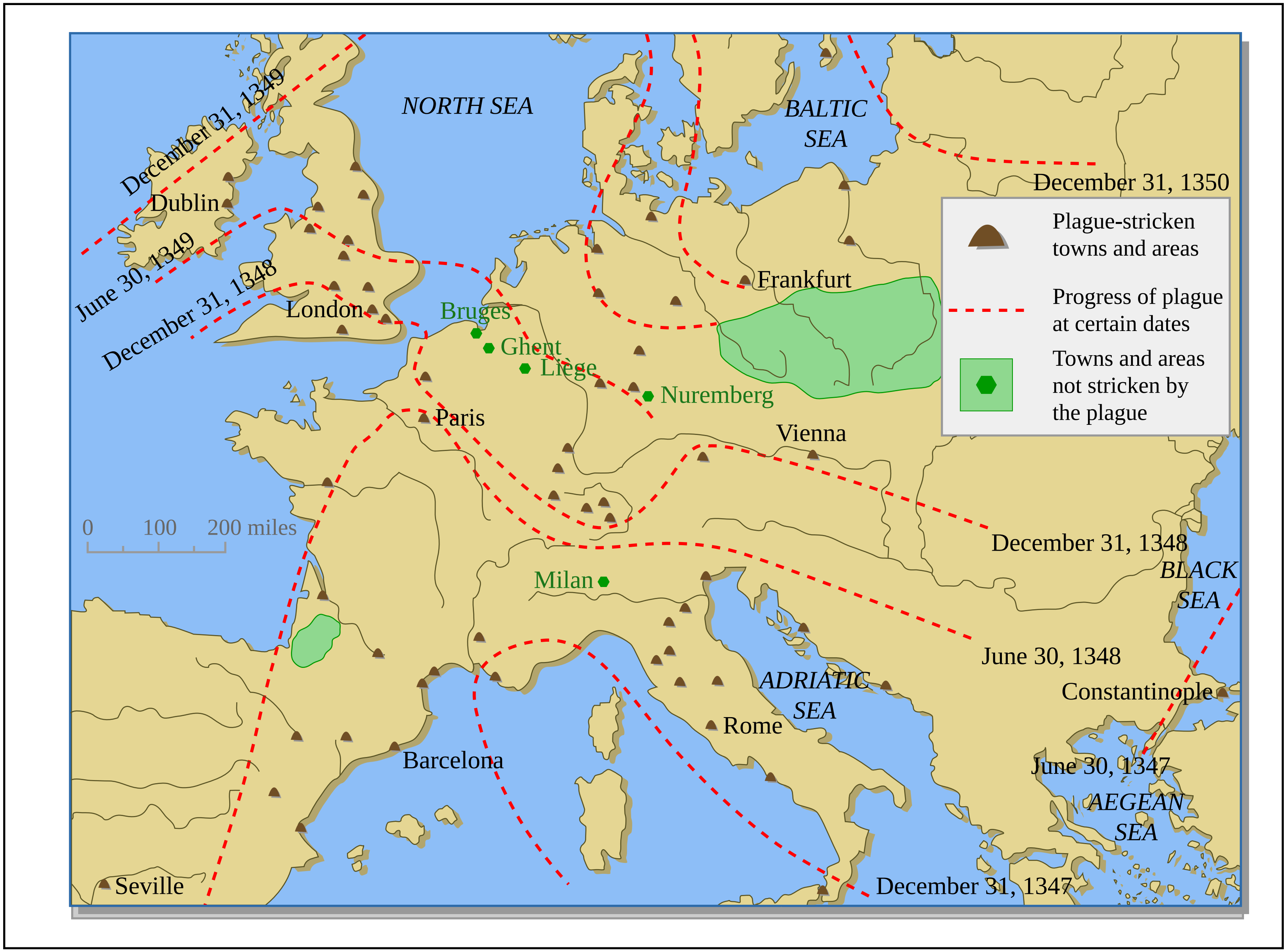
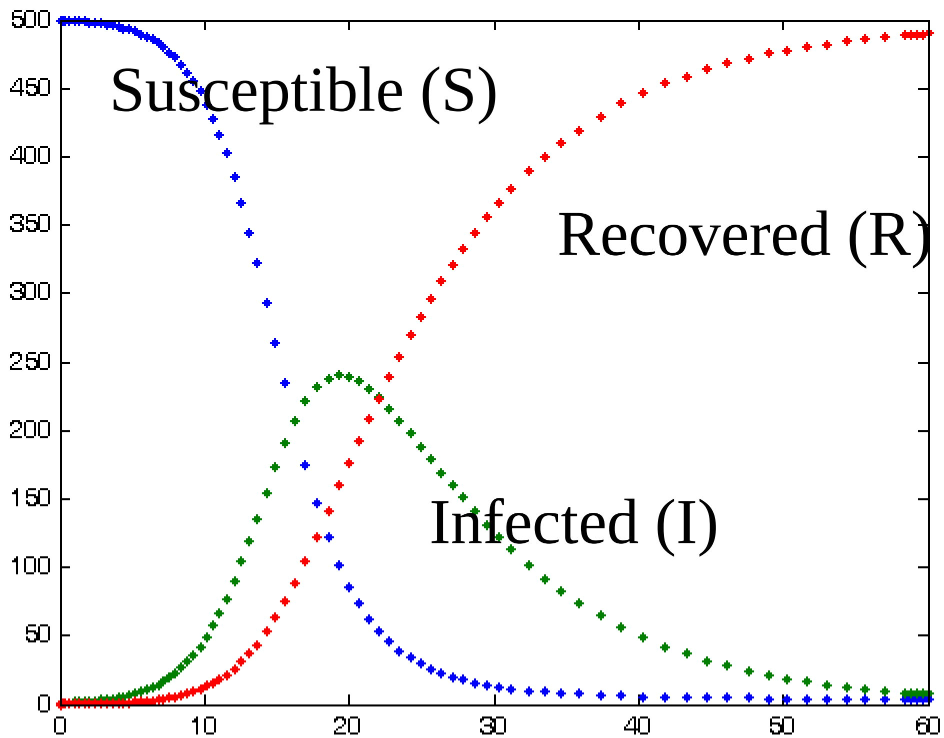


Image by MIT OpenCourseWare.

Ralph's World Civilizations, Chapter 13

SIR Model



Values of R_0 of well-known infectious diseases[1]

Disease	Transmission	R_0
Measles	Airborne	12–18
Pertussis	Airborne droplet	12–17
Diphtheria	Saliva	6–7
Smallpox	Social contact	5–7
Polio	Fecal-oral route	5–7
Rubella	Airborne droplet	5–7
Mumps	Airborne droplet	4–7
HIV/AIDS	Sexual contact	2–5[2]
SARS	Airborne droplet	2–5[3]
Influenza (1918 pandemic strain)	Airborne droplet	2–3[4]e
H1N1	**Airborne droplet**	**1.5?**

$$\dot{S} = -\beta IS$$

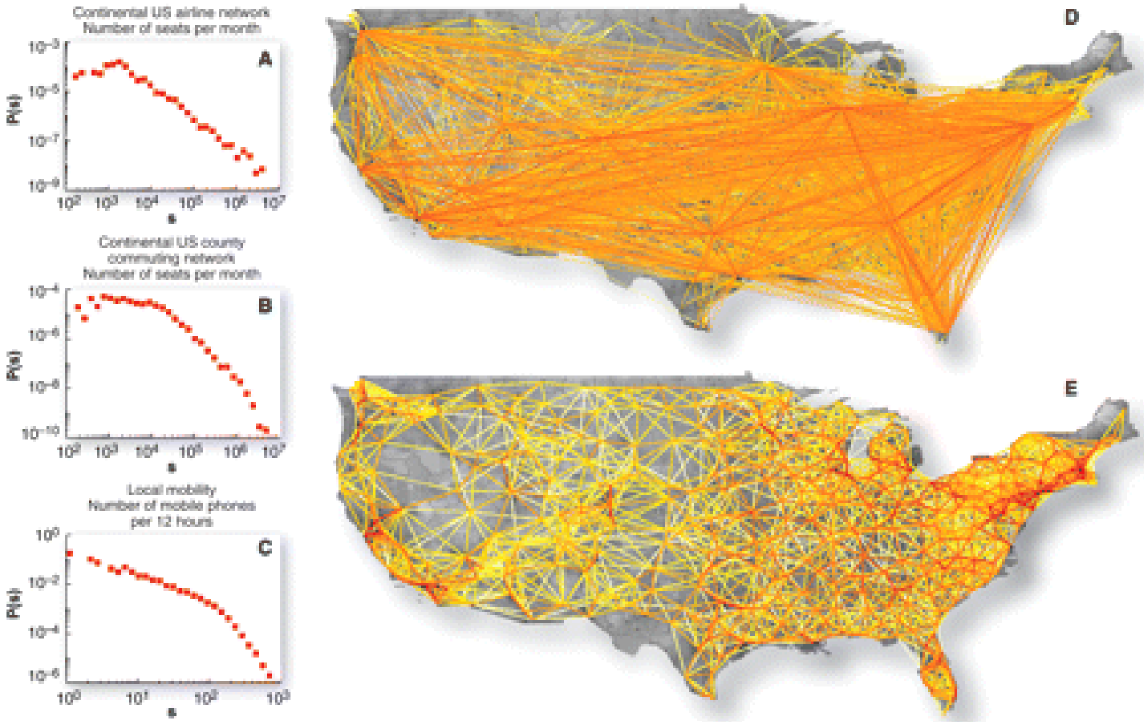
$$\dot{I} = \beta IS - \nu I$$

$$\dot{R} = \nu I$$

$$R_o = \beta / \nu \text{ "Basic Reproduction Number"}$$

http://en.wikipedia.org/wiki/Compartmental_models_in_epidemiology
http://en.wikipedia.org/wiki/Basic_reproduction_number

Transport Networks Cause Fast Diffusion

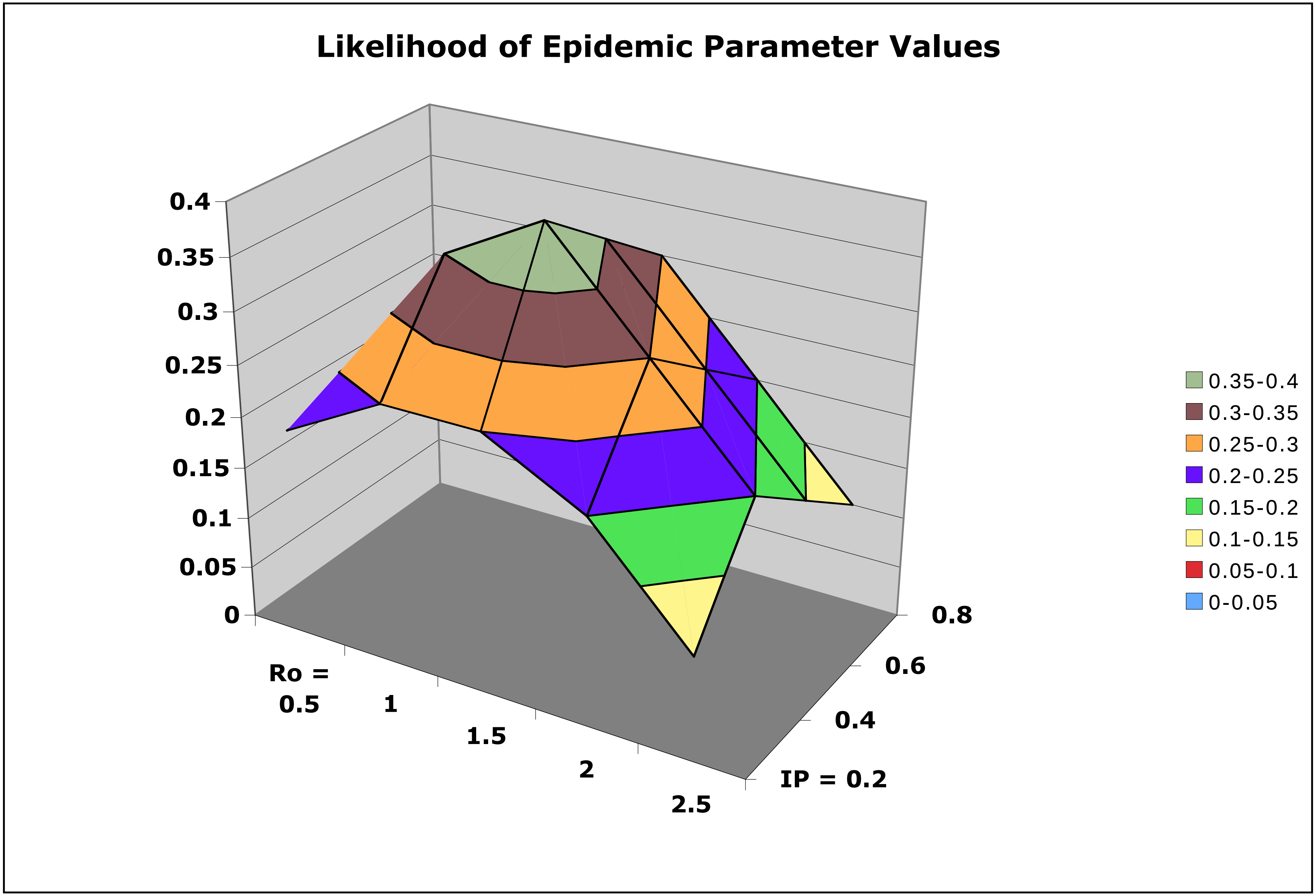


US air travel

US local commuting

Global air travel

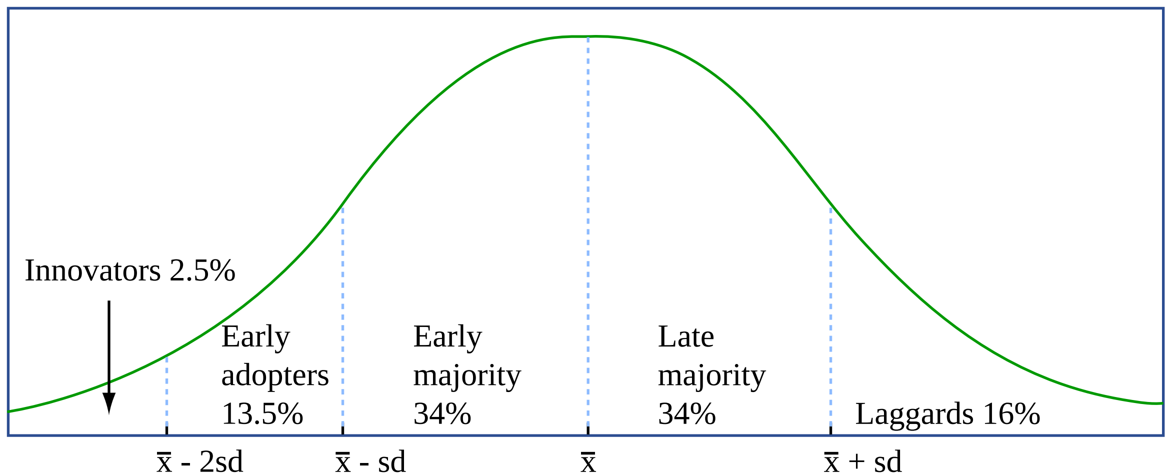
Using Data and Model to Find Parameters



Adoption of Innovations - Rogers

Innovativeness and Adopter Categories

Adopter categorization on the basis of innovativeness

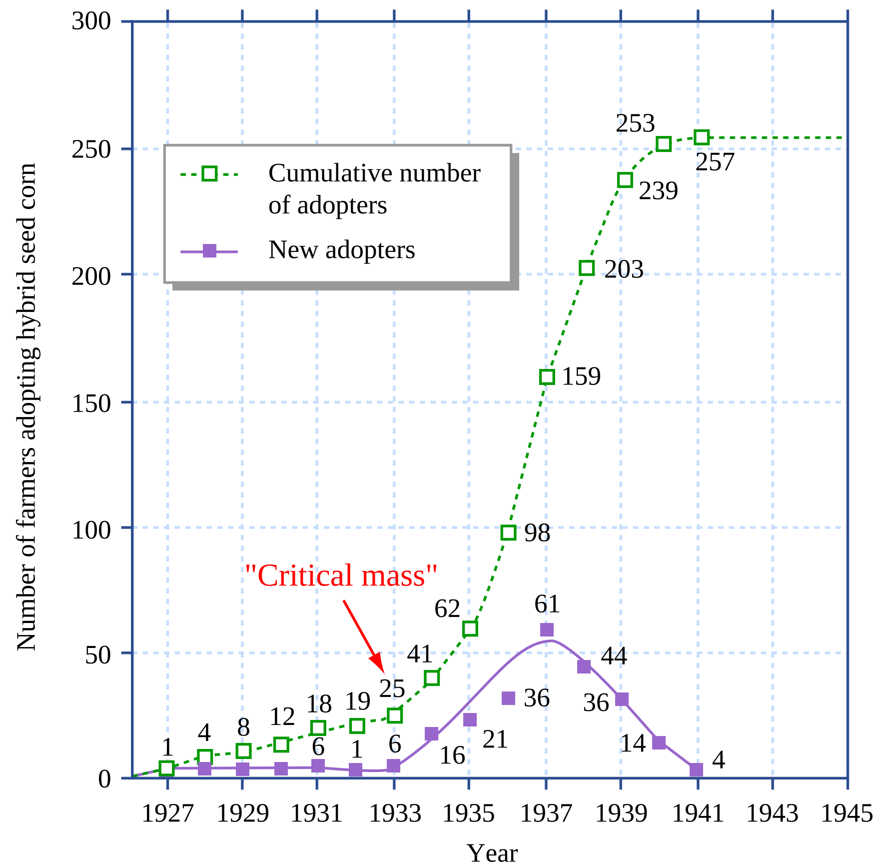


The innovativeness dimension, as measured by the time at which an individual adopts an innovation or innovations, is continuous. The innovativeness variable is partitioned into five adopter categories by laying off standard deviations (sd) from the average time of adoption ($\bar{x}$).

Image by MIT OpenCourseWare.

Basic idea: later adopters wait until more have adopted first.
Gives rise to threshold models of diffusion and percolation.

Adoption of Hybrid Corn - Rogers



The number of new adopters each year, and the cumulative number of adopters, of hybrid seed corn in two Iowa communities

What's Interesting About Random Networks

- They represent one extreme of networks
  - Another is regular structures like grids or arrays
  - Another is "designed" networks with rational but not necessarily regular structure
- They can be analyzed mathematically ("light's better")
- The non-randomness of other networks can sometimes be measured by comparing metrics with random networks of similar size and density
- Some real networks are more random than one would imagine
- Some random networks harbor non-random properties

Basic Theory

- Network has n nodes
- A pair of nodes is linked (both ways) with probability p
- The number of links in the network $m = pn(n-1)/2$
  - Some fraction of the number if all nodes were linked, counting each pair of nodes once
- The average nodal degree = $z = 2m/n = pn$
- The clustering coefficient $C = pr(2\ neighbors\ linked) = pr(any\ pair\ linked) = p = z/n$
- For given z, C goes down as the size of the network grows
- For many properties "*P*" we find that *P* "suddenly appears" when tracked according to some network parameter like z
  - "Sudden appearance" is usually called a phase transition
  - The most common example *P* is connectedness

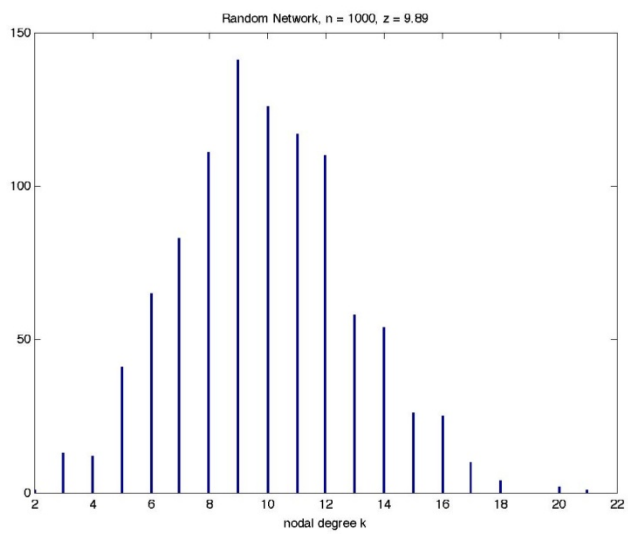
Subgraph Shapes in Finite Random Networks in Matlab

- When $z = 1$, about one third of the nodes are isolated and have $k = 0$ while another third have $k = 1$, implying lots of linked pairs of nodes
- Clusters, if any, have $z \sim 2$ and contain the last third
  - Small stars, chains, trees
  - Few closed loops
- To get a big connected cluster, we need $z > 1$ for the graph as a whole and $z > 2$ for a connected cluster because z of a tree is ~2
  - For a tree, $m = n - 1, z = 2m/n, \therefore z \approx 2$

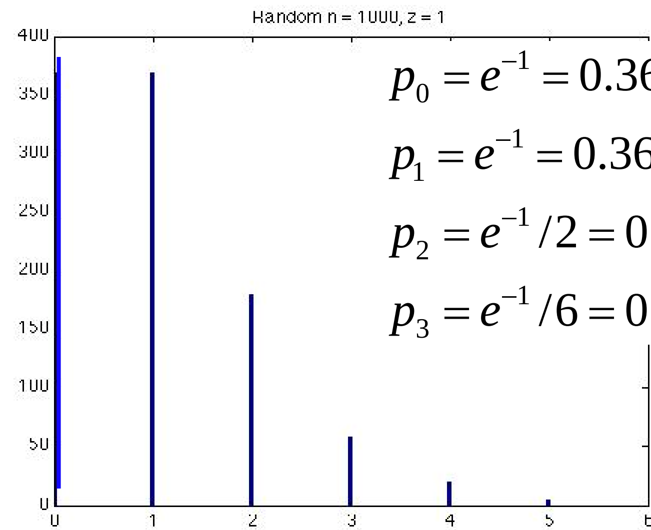
Degree Distribution of ER Random Network

- For large *n* the degree distribution is Poisson, and *z=np* is the only adjustable parameter

$$p_k = \binom{n-1}{k} p^k (1-p)^{n-k-1} \cong e^{-z}\frac{z^k}{k!} \text{ if } n \to \infty$$



z = 10



$$p_0 = e^{-1} = 0.3679$$
$$p_1 = e^{-1} = 0.3679$$
$$p_2 = e^{-1}/2 = 0.1359$$
$$p_3 = e^{-1}/6 = 0.0453$$

z = 1.006

- This looks roughly Gaussian for large *z*, highly peaked for small *z*
- Standard deviation $\sigma = \sqrt{z}$

Average Path Length and Network Diameter

- Typical node has *z* neighbors
- Each of them has *z* neighbors (assumes somewhat tree-like structure, true when *z* is not much bigger than 1)
- At distance *l* there are ~ z^l neighbors (if *z* is small so that the network is mainly tree-like)
- If *d* = the shortest distance all the way across a network of *n* nodes, then $z^d \sim n$ and $d \sim ln(n)/ln(z)$
- Average path length < diameter so $l \sim ln(n)/ln(z)$
- Exact formula for APL:

$$<\ell>= \frac{\ln(n)-\gamma}{\ln(z)}+0.5$$

$$\gamma = \text{ Euler's number } = 0.5771$$

Average path length in random networks
Agata Fronczak, Piotr Fronczak, and Janusz A. Hołyst
PHYSICAL REVIEW E 70, 056110 (2004)

Percolation and Cascades

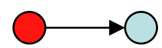
- Both terms are ~synonymous with emergence of a "giant cluster"
  - In an infinitely large random network, the size of a connected cluster is a non-zero % of the total number of nodes
  - In a finite network, the cluster size is comparable to the size of the network
- Giant clusters appear if the network is dense enough
- The proven threshold for E-R is $z = 1$

Percolation, Cascades, Rumors

- A network consists of nodes that can be “flipped” from their initial state (off) to another state (on) depending on their “vunerability”
- A “seed” node (or in some models, a set of seed nodes) is arbitrarily switched from off to on
- Subsequently, other neighboring nodes may flip, depending on model assumptions
- The cascade will not permeate the whole network unless the network “percolates” or is connected with probability = 1
- Even if it is connected, it still may not percolate

Vulnerability and Stability

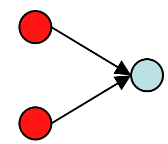
- A node is "vulnerable" if one flipped neighbor can flip it



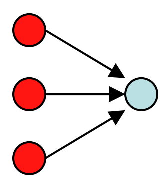
$k \le K^*$

- A stable node is "first order stable" if two flipped neighbors can flip it

$K^* < k \le 2K^*$



- A stable node is "second order stable" if three flipped neighbors can flip it
- etc



$2K^* < k \le 3K^*$

Percolation Theory for Sparse Random Graphs

- Derived by Newman and others using generating functions
- Recreates and extends the Molloy-Reed criterion
- Extended by Watts
- Assumes graphs (or vulnerable subgraphs) are trees and networks are of infinite size

All nodes vulnerable $\sum_{k=0}^{\infty} k(k-1)p_k = z$ Molloy-Reed criterion

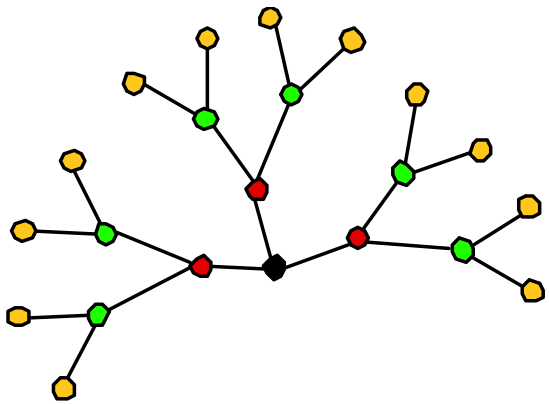
Vuln with *pr = b* $b\sum_{k=0}^{\infty} k(k-1)p_k = z$

Vuln is fct of *k* $\sum_{k=0}^{\infty} k(k-1)\rho_k p_k = z$

Watts rumor cascade model:

$$\rho_k = \begin{cases} 1 \text{ for } k \leq K* \\ 0 \text{ for } k > K* \end{cases}$$

Example Percolation on a Tree



z_i = excess degrees at step i
n_i = ith neighbors

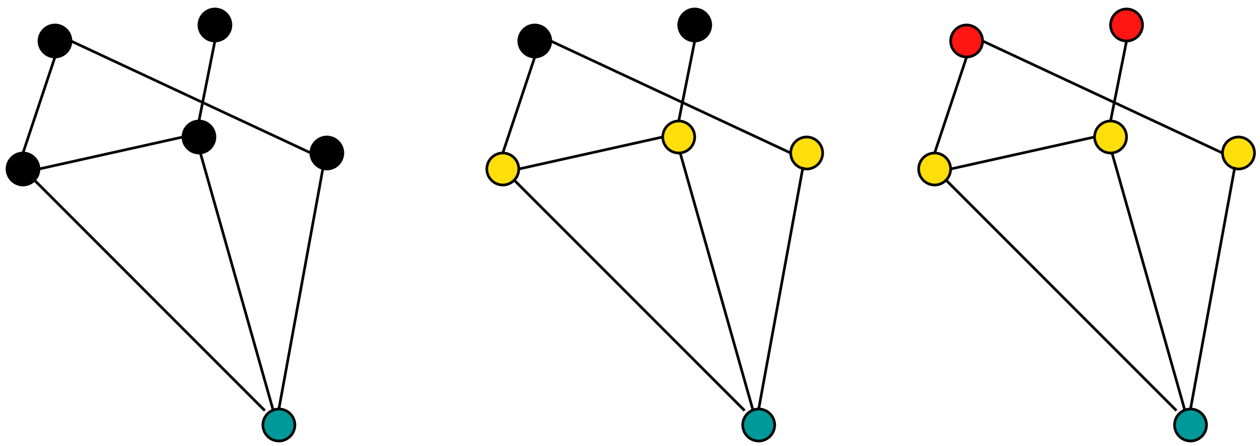
$z_1=3 \quad n_1=3$

$z_2=2 \quad n_2=z_2*n_1=2*3=6$

$z_3=2 \quad n_3=z_3*n_2=z_3*z_2*n_1=2*2*3=12$

$$\text{For E - R, avg excess degree of a neighbor} = z - 1$$

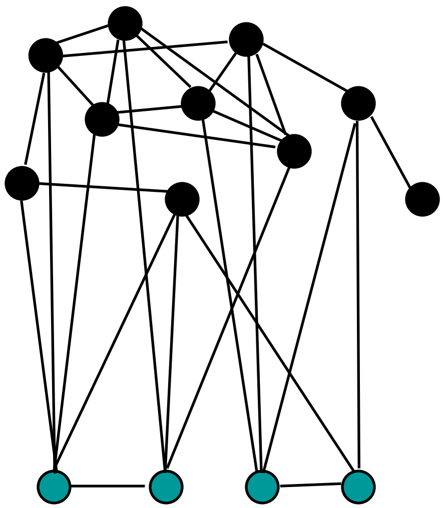
Single Node Seed, No Threshold



Seed First Step Second Step

Number of nodes hit on first step = number of edges out from seed = *Sz*

Multi-Node Seed, Threshold

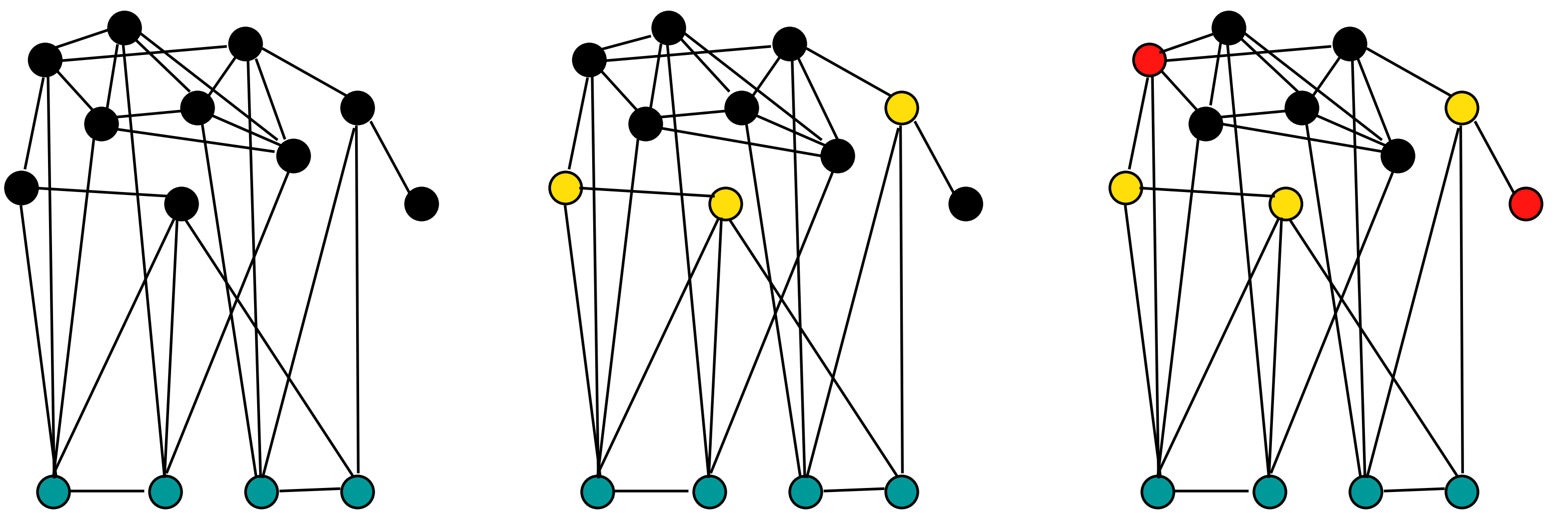


Number of nodes hit < number of edges out because some nodes are hit multiple times, allowing stable nodes to be flipped

Two Steps, Multi-Node Seed, Threshold

$1 \le k \le K^*$: flip if $\ge$ 1 neighbor flips

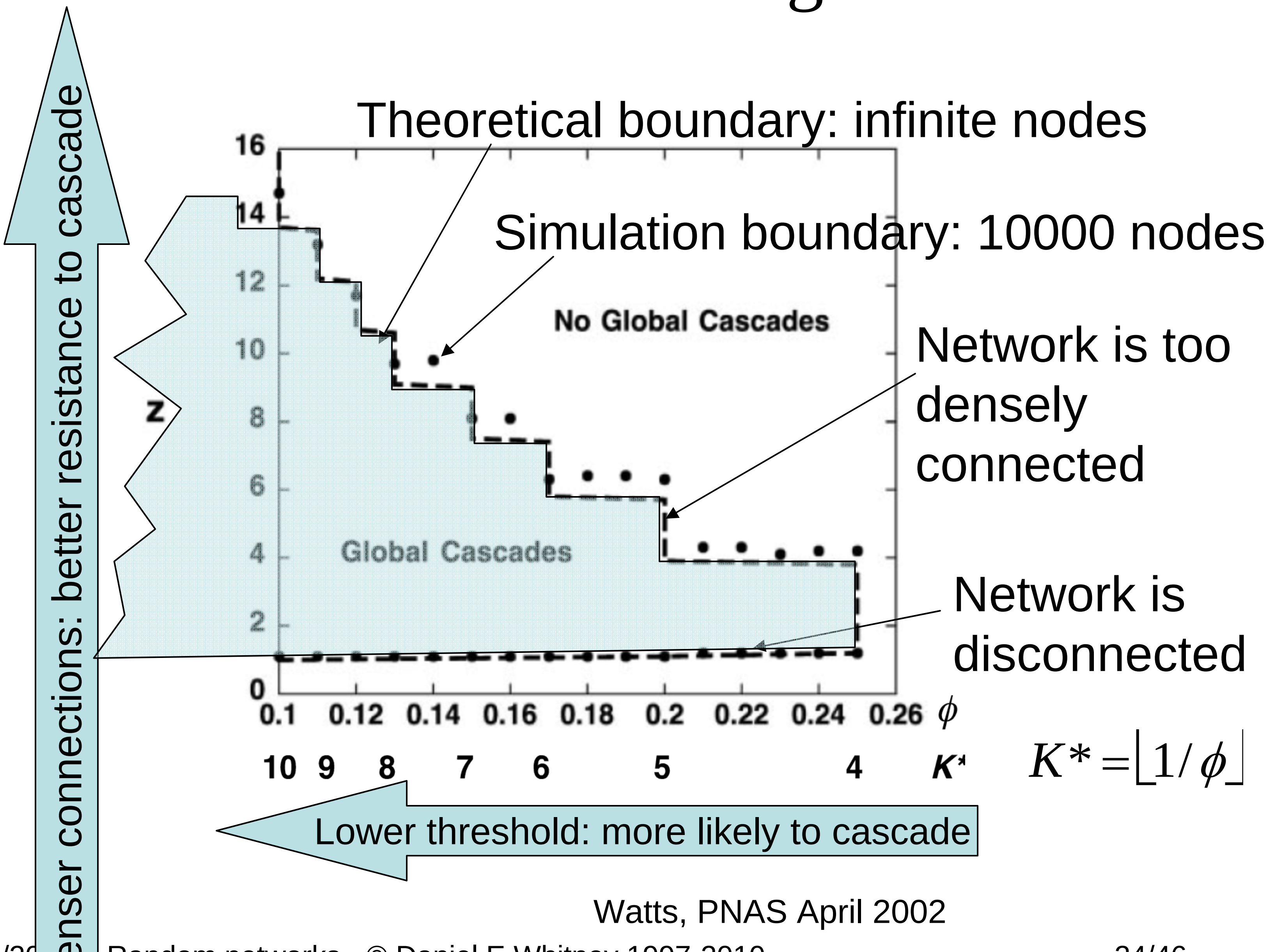
$K^* + 1 \le k \le 2K^*$: flip if $\ge$ 2 neighbors flip



Seed First Step Second Step

K* = 4

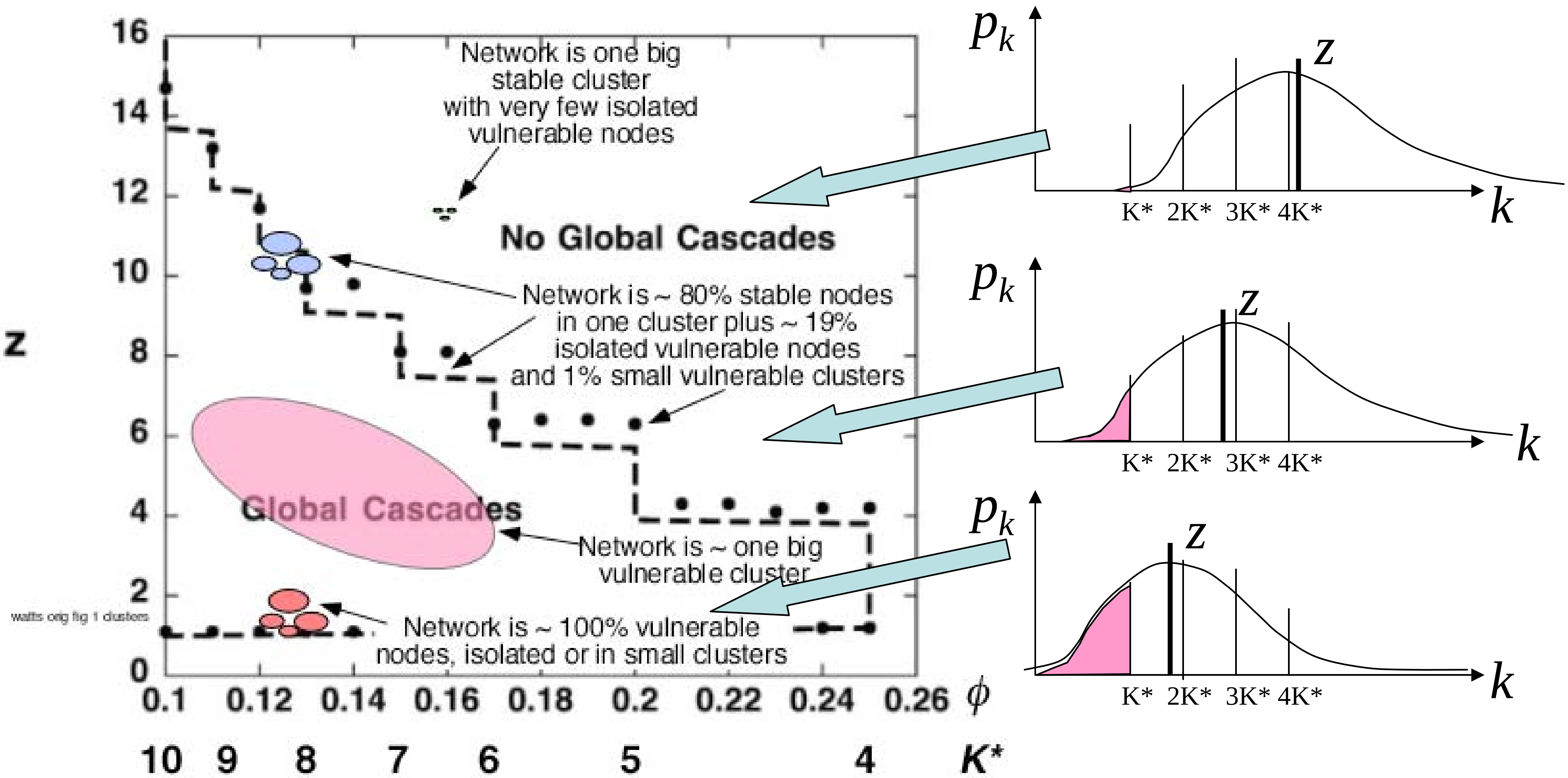
Watts' Cascade Diagram



$$K^* = \lfloor 1/\phi \rfloor$$

Watts, PNAS April 2002

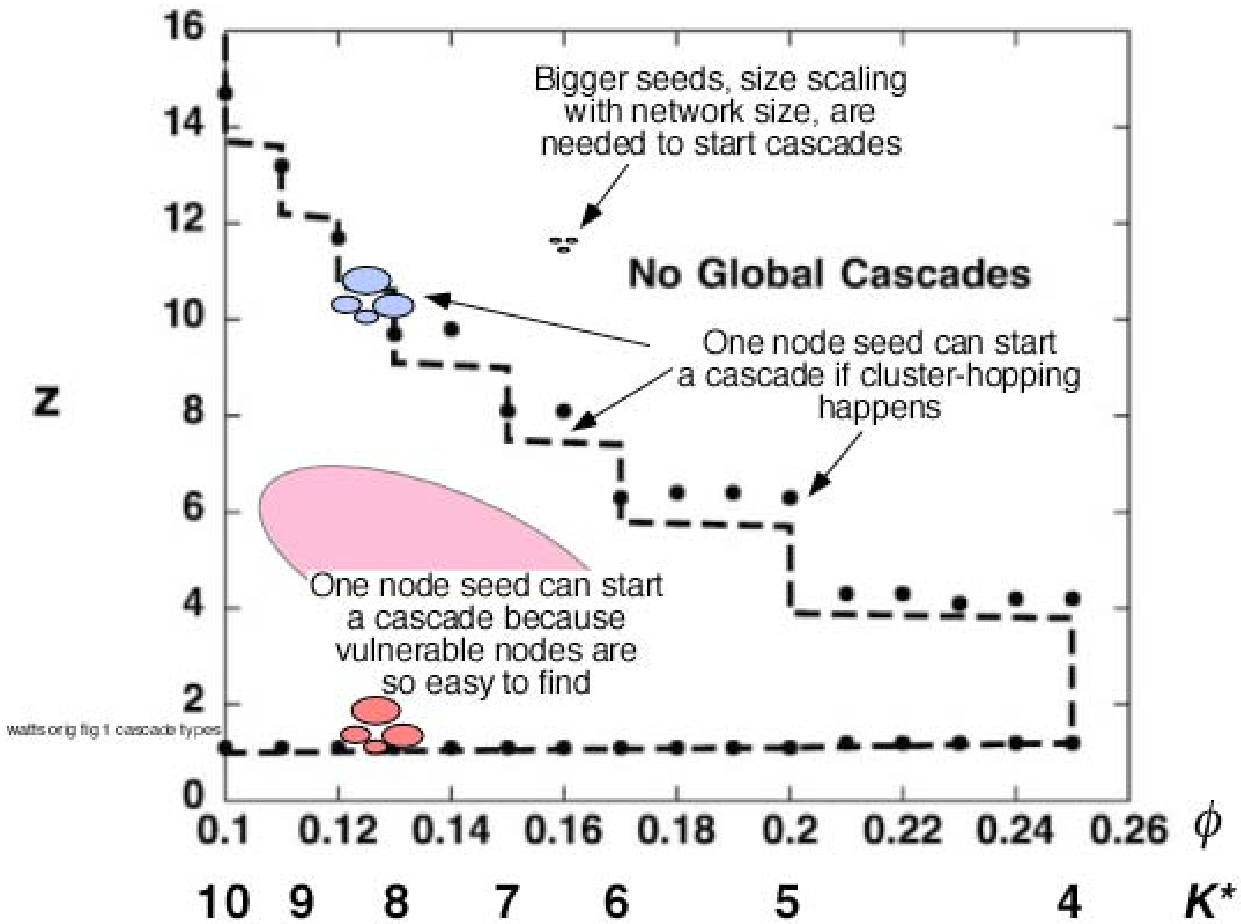
Vulnerable Clusters in Finite E-R Networks



Watts Theory and Simulations

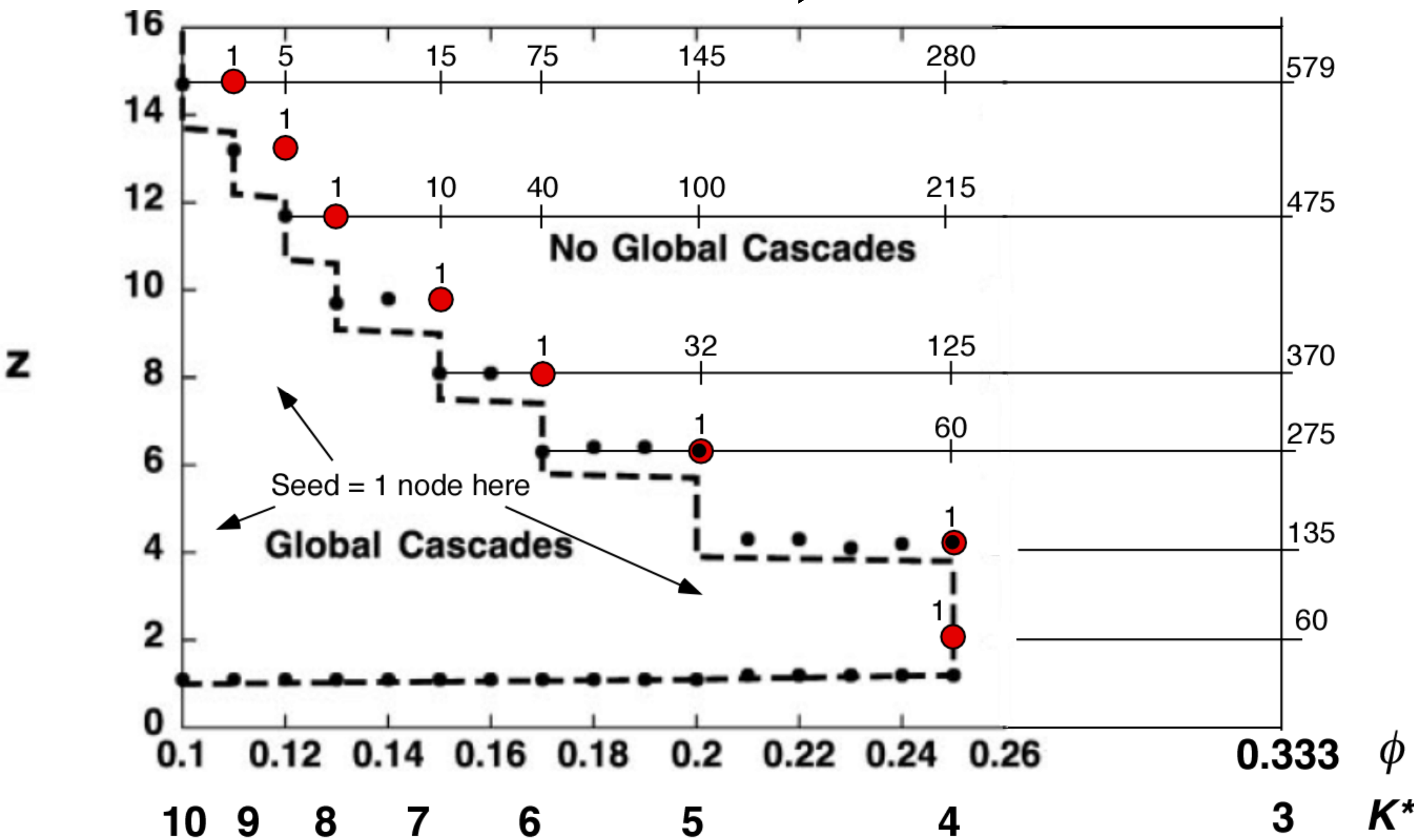
- Theory
  - Global cascades region: “small” seed can start a global cascade
  - No global cascades region: “small” seed cannot start a global cascade
- Simulations on network with 10000 nodes
  - Global cascades region: seed of one node can start a global cascade
  - No global cascades region: seed of one node cannot start a global cascade

Cascades in Finite E-R Networks Can Happen in the No Global Cascades Region

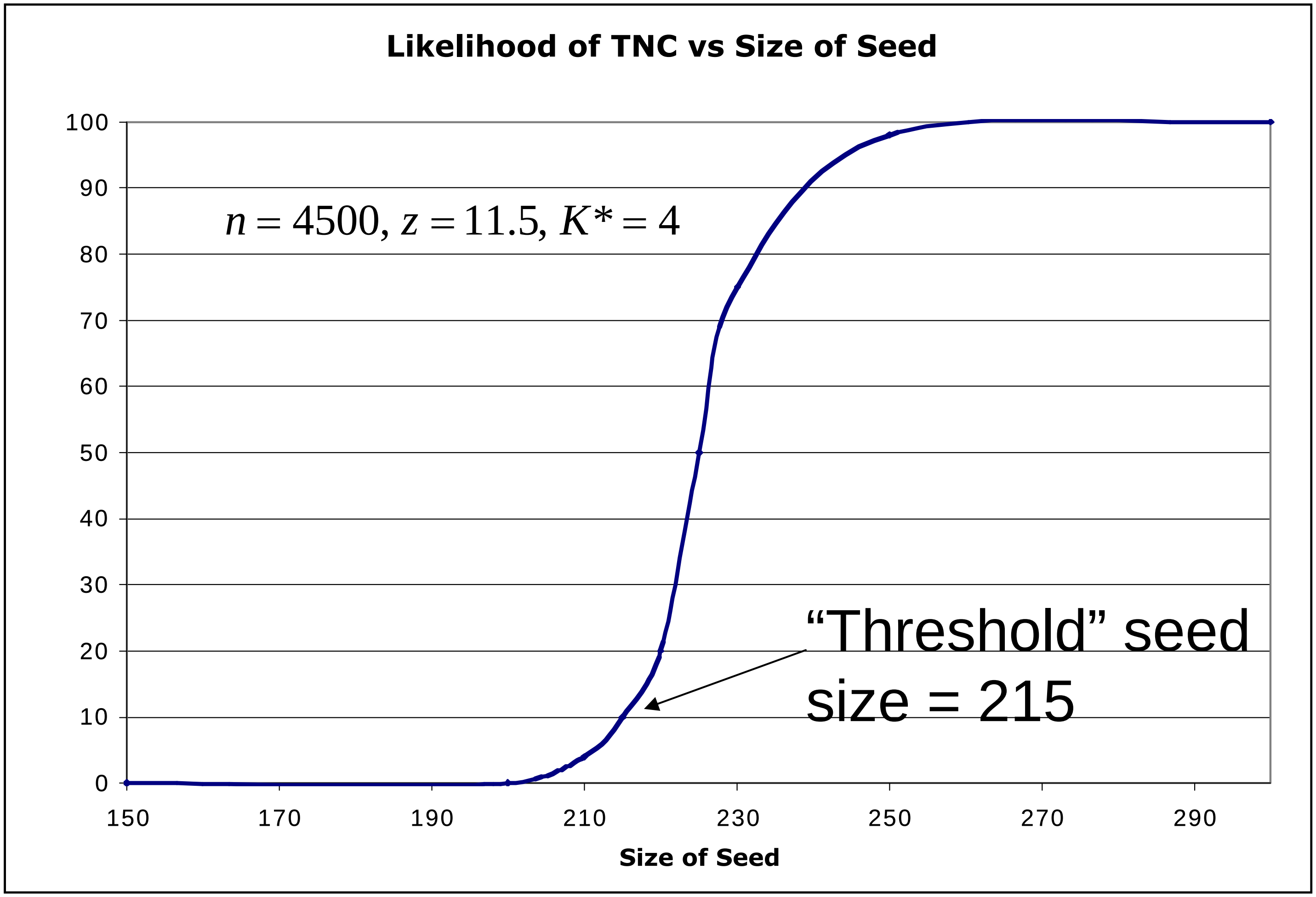


D. E. Whitney, ÒDynamic theory of cascades on finite clustered random networks with a threshold ruleÓ *Physical Review E*. E **82**, 066110 (2010)

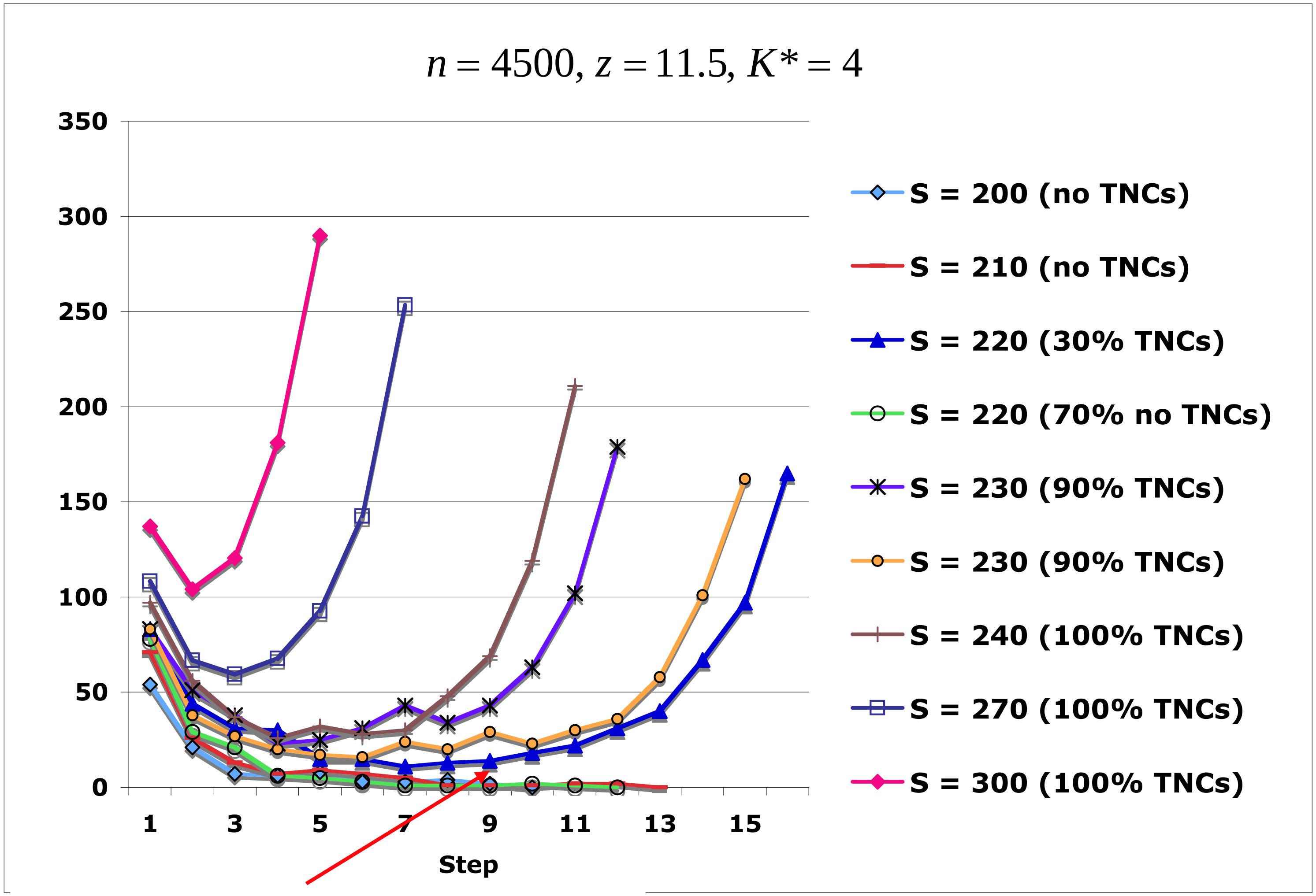
Simulations: Necessary Seed Sizes (n = 4500)



Simulations: Threshold Seed Size - A Phase Transition



Typical Cascade Trajectories Throughout Transition Range of Seed Size

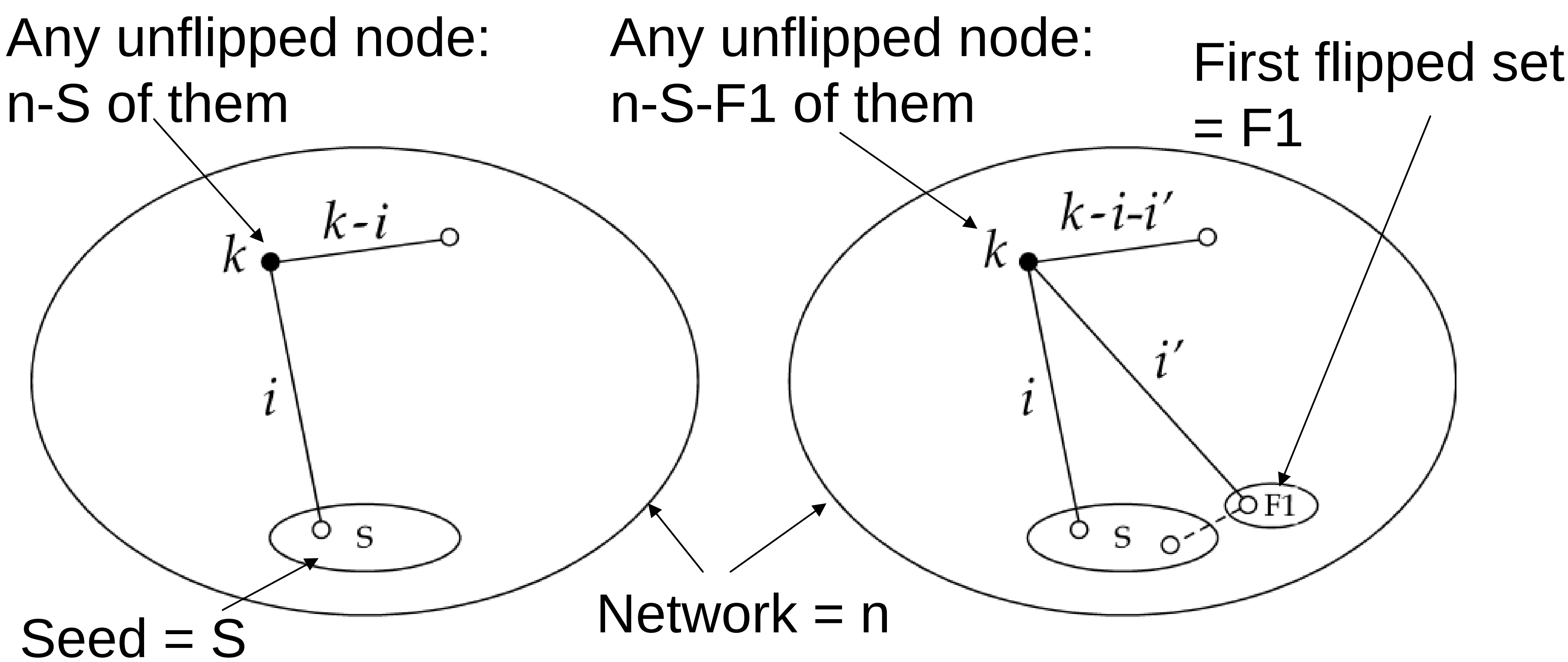


“Near death” phenomenon

Theory

$$p_k = \sum_{i=0}^{k} p_S(i,S)\,p_{nS}(k-i,n-S)$$

$$p_k = \sum_{i=0}^{k} \binom{S}{i} p^i (1-p)^{S-i} \binom{N-1-S}{k-i} p^{k-i} (1-p)^{N-1-S-(k-i)}$$



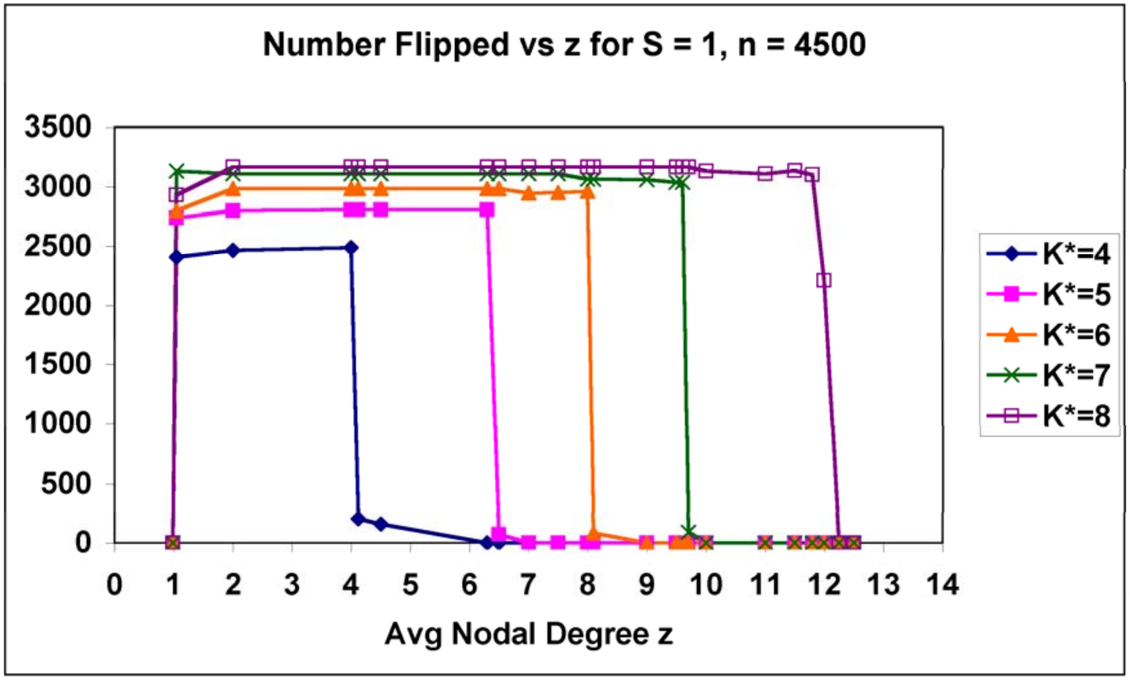
Theory Step 2

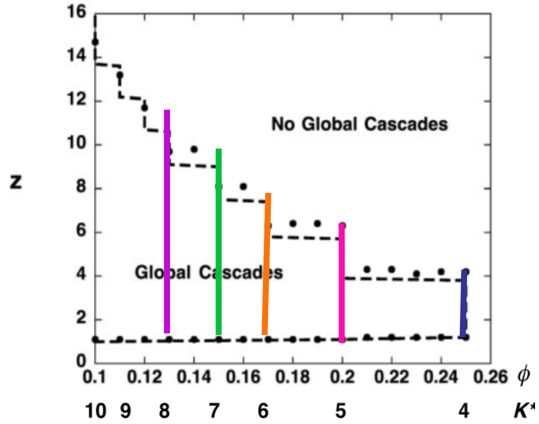
$$p_k = \sum_{i=0}^{k} \sum_{i'=0}^{k-i} \binom{S}{i} p^i (1-p)^{S-i} \binom{F1}{i'} p_{F1}^{i'} (1-p_{F1})^{F1-i'}$$
$$\times \binom{n-S-F1-1}{k-i-i'} p_{nSF1}^{k-i-i'} (1-p_{nSF1})^{n-S-F1-1-(k-i-i')}$$

$p_{F1} = z_{F1}/n$ reflects available edges from *F1*

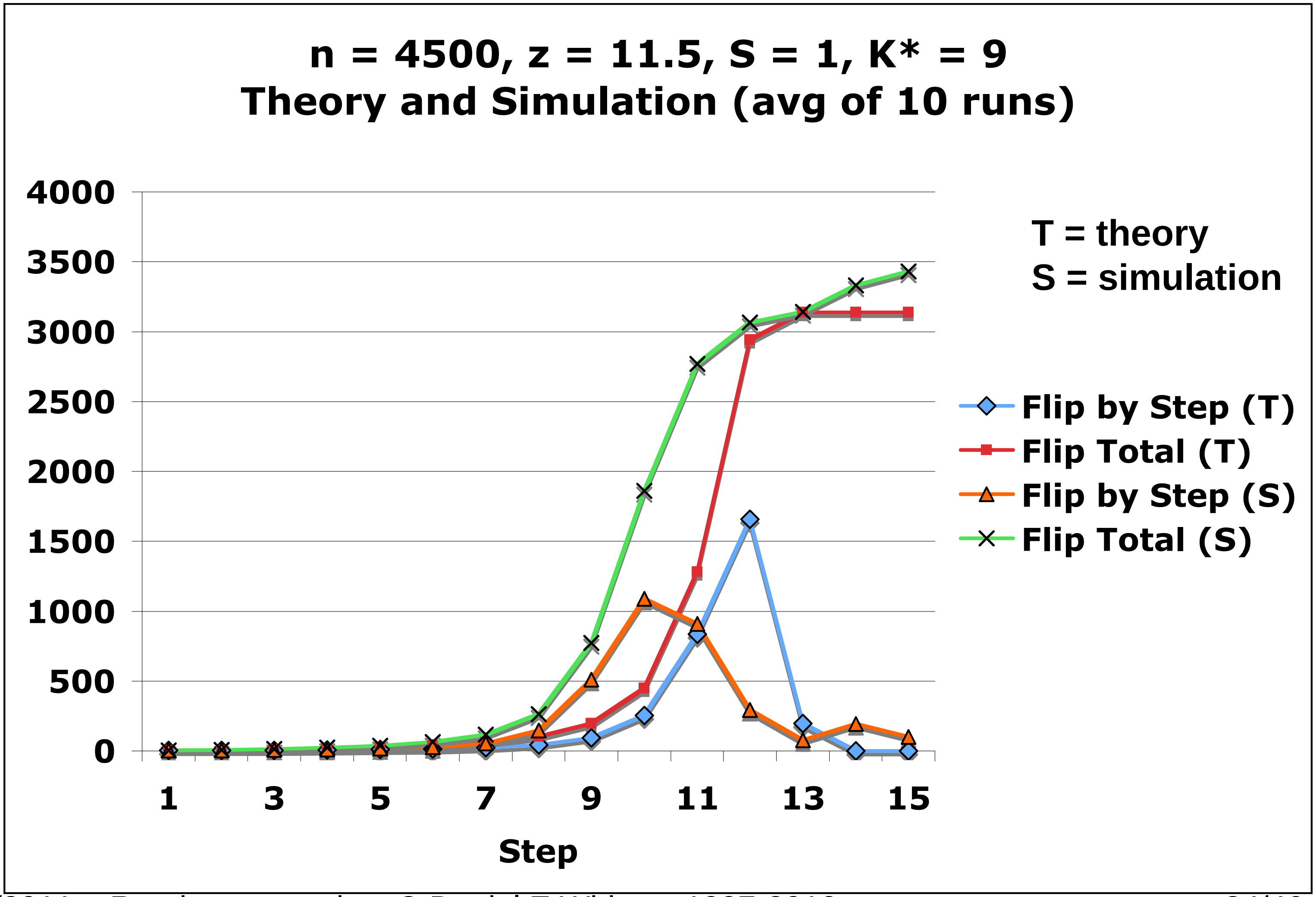
p_{nSF1} reflects larger *p* of unflipped nodes

Theory: Cascades in "Global Cascades" Region - Seed = 1 Node

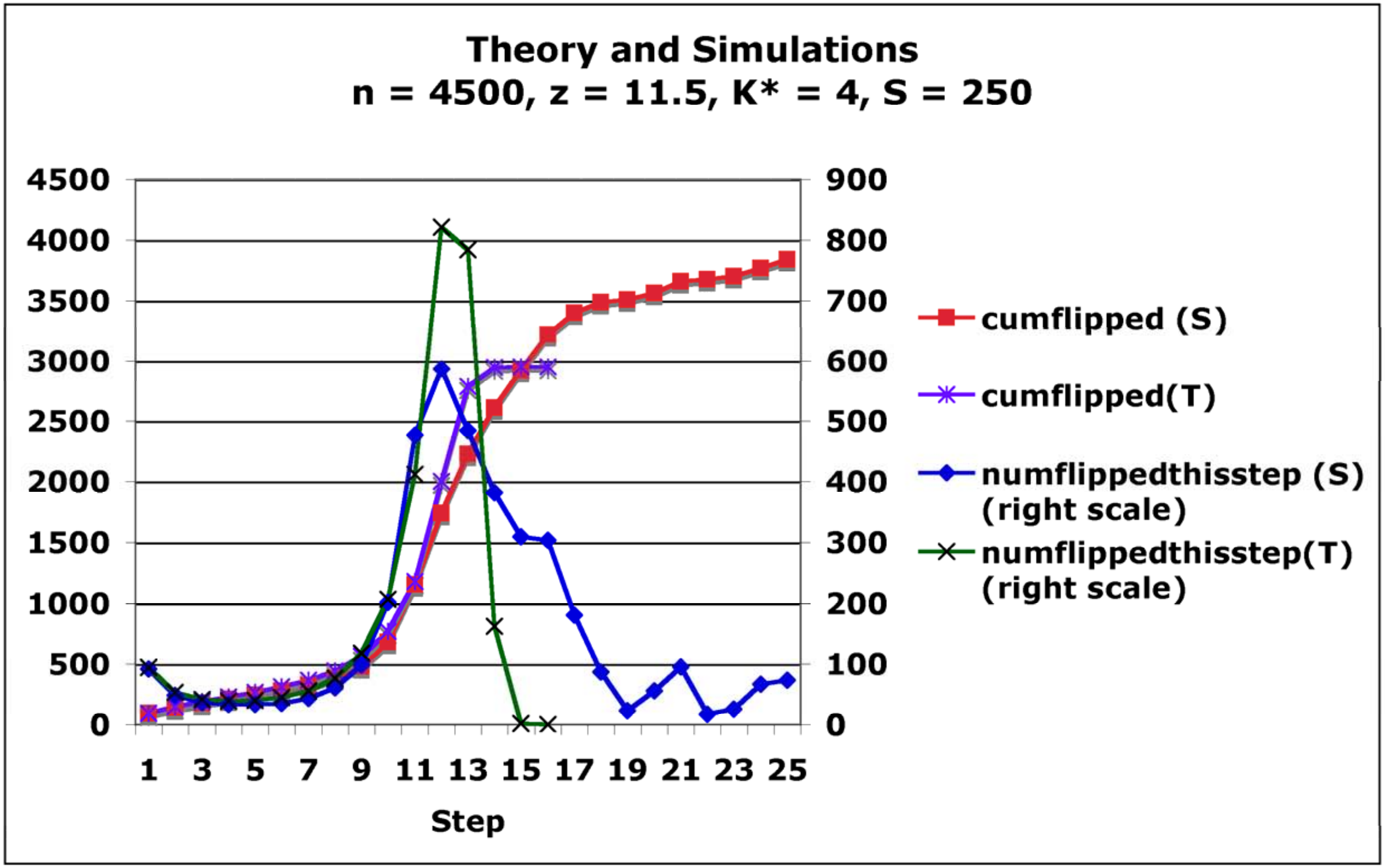




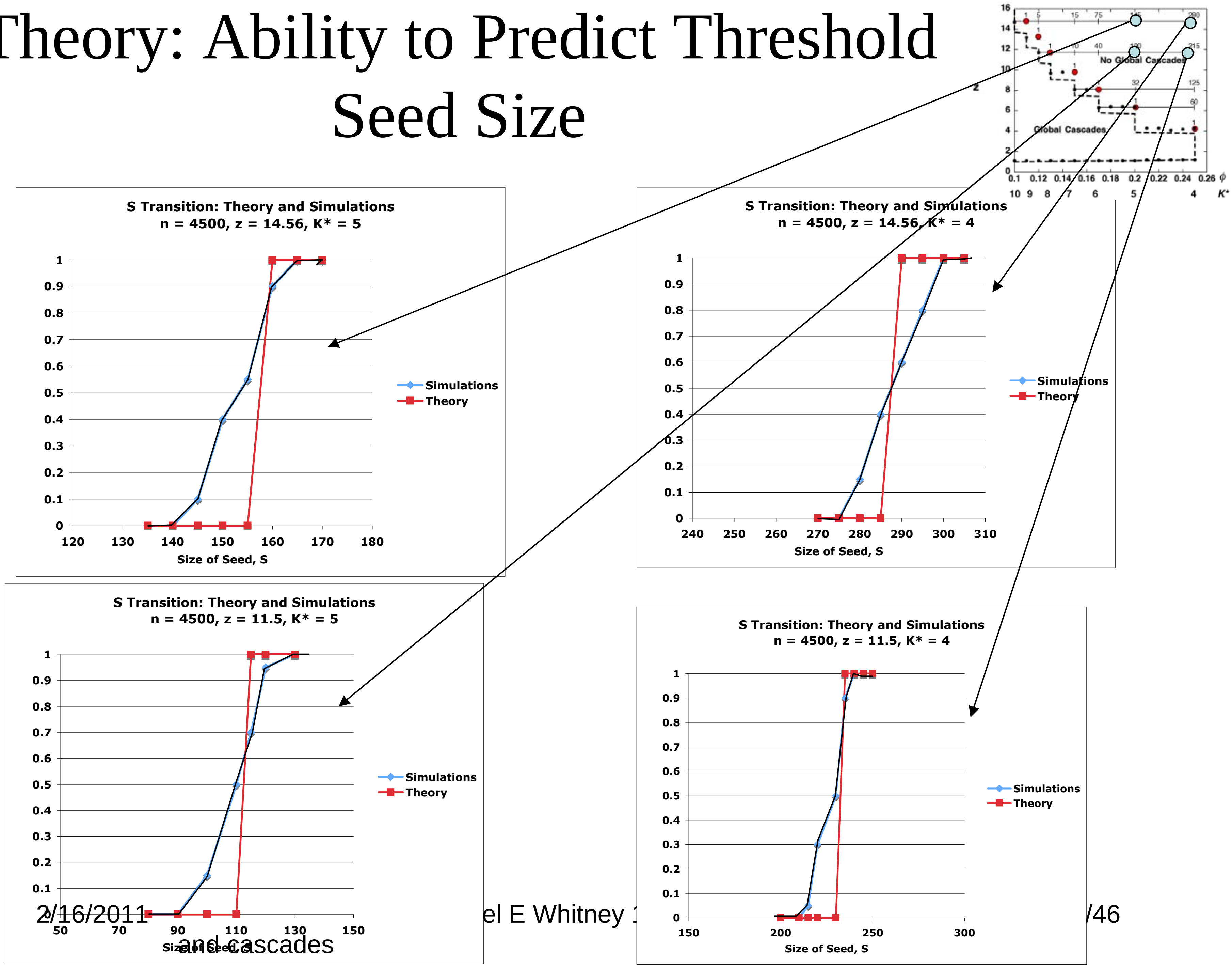
Theory and Simulations: Cascade in “Global Cascades” Region



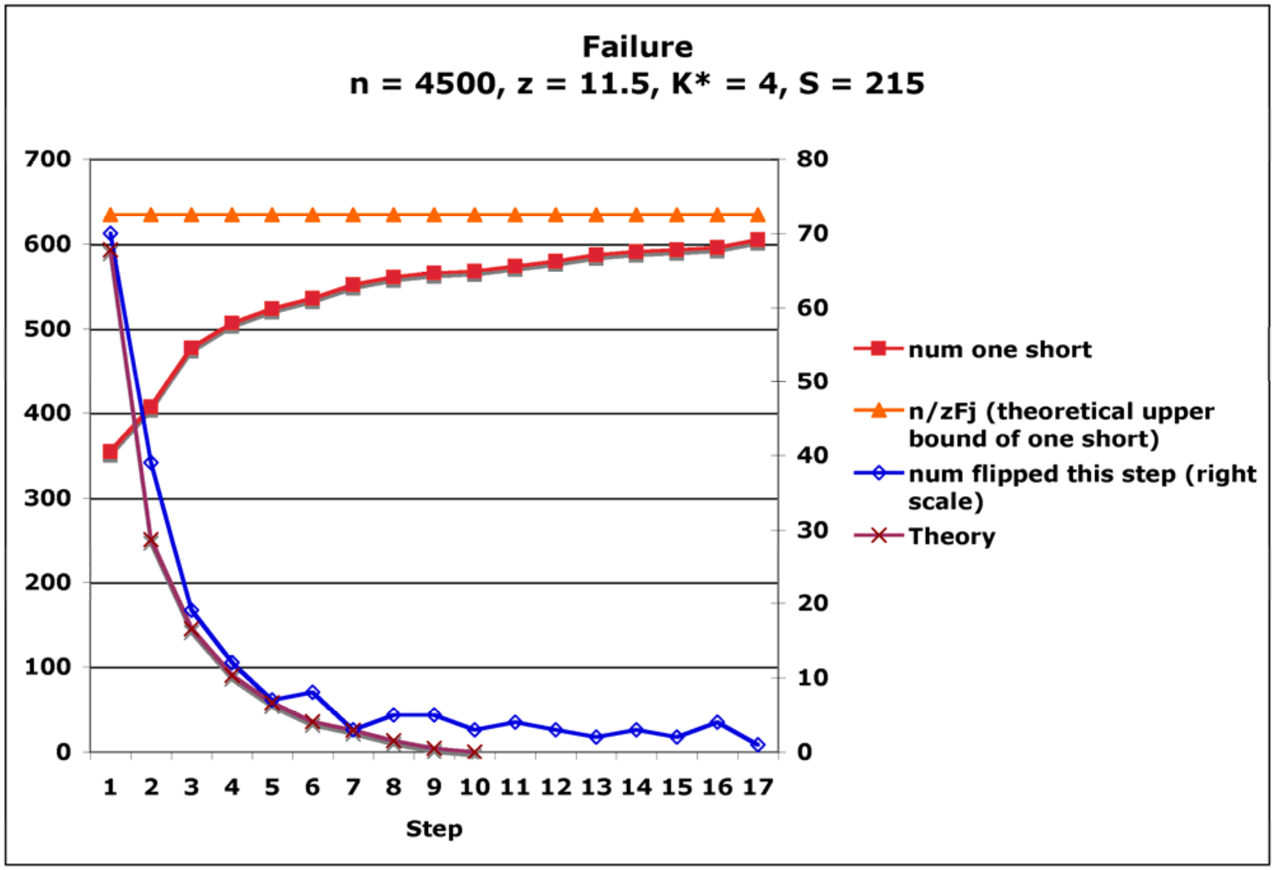
Theory and Simulations: A Cascade in "No Global Cascades" Region



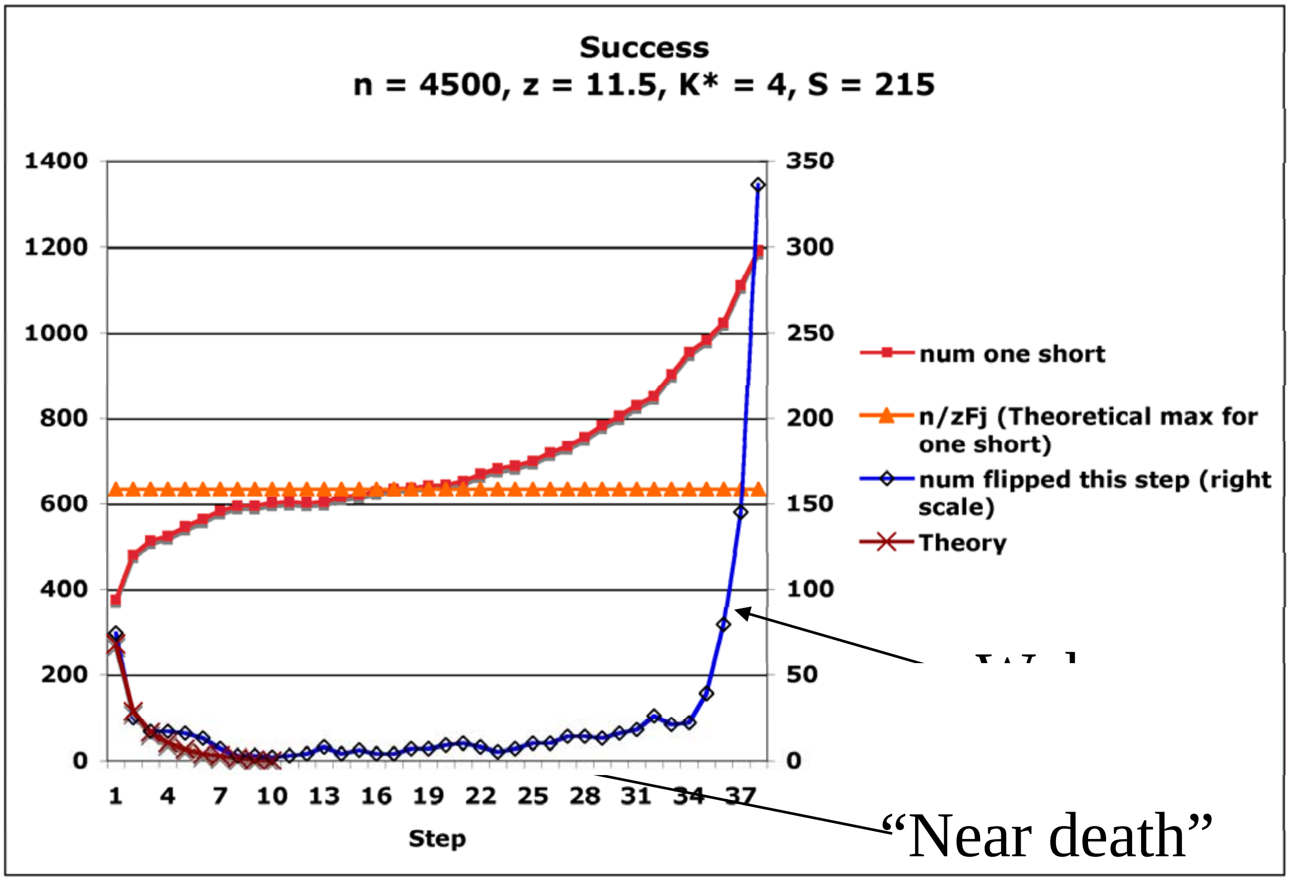
Theory: Ability to Predict Threshold Seed Size



At S = 215, Failure Most of the Time



Occasionally, Success: Why?



Cause of Wake-up After Near Death

- Caused by a critical mass phenomenon
- Near death only a few nodes flip on each step
- At most they can hit one node each since, for so few nodes, the likelihood of multiple hits is about zero
- So only nodes that are one hit short of flipping have any chance to flip during this phase
- This chance is proportional to how many one-shorts there are on any step and how many net edges *Fj* has
- This population is growing but at the same time the number of flippers is falling

Derivation of Critical Mass

$$pr(\text{a node in the network links to } Fj) = \frac{Fj\, z_{Fj}}{n}$$

$$\#\,\text{nodes with edges to } Fj = Fj\, z_{Fj}$$

$$= \#\,\text{nodes that will be hit by } Fj$$

$$\text{fraction of these that will flip} =$$

$$\text{fractional representation of one short in the network} = \frac{N_{OS}}{n}$$

$$\text{number of nodes that } Fj \text{ flipped nodes will flip} = \frac{N_{OS} Fj\, z_{Fj}}{n}$$

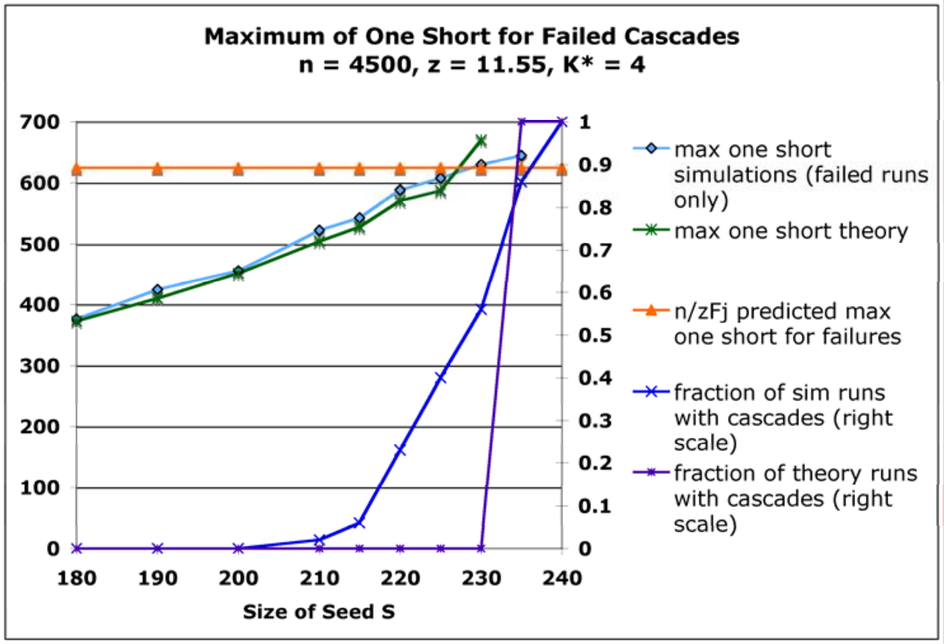
If each node in Fj flips one node, the cascade is self-sustaining.

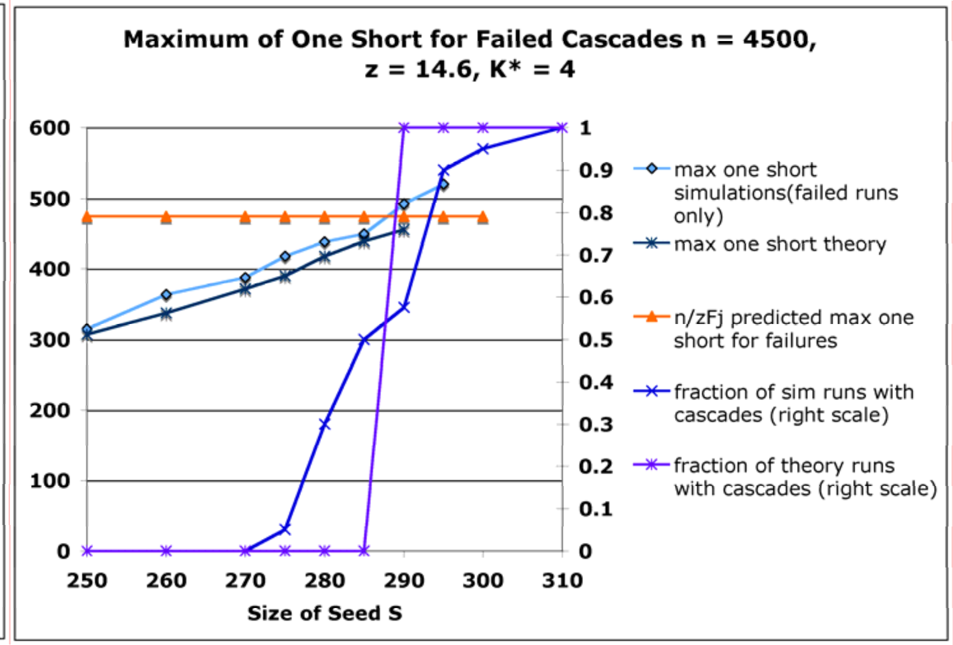
$$\text{So } 1 = \frac{N_{OS} z_{Fj}}{n}$$

$$\text{or } N_{OS} = n / z_{Fj}$$

$$\text{or } Fj * N_{OS} = Fj * n / z_{Fj}$$

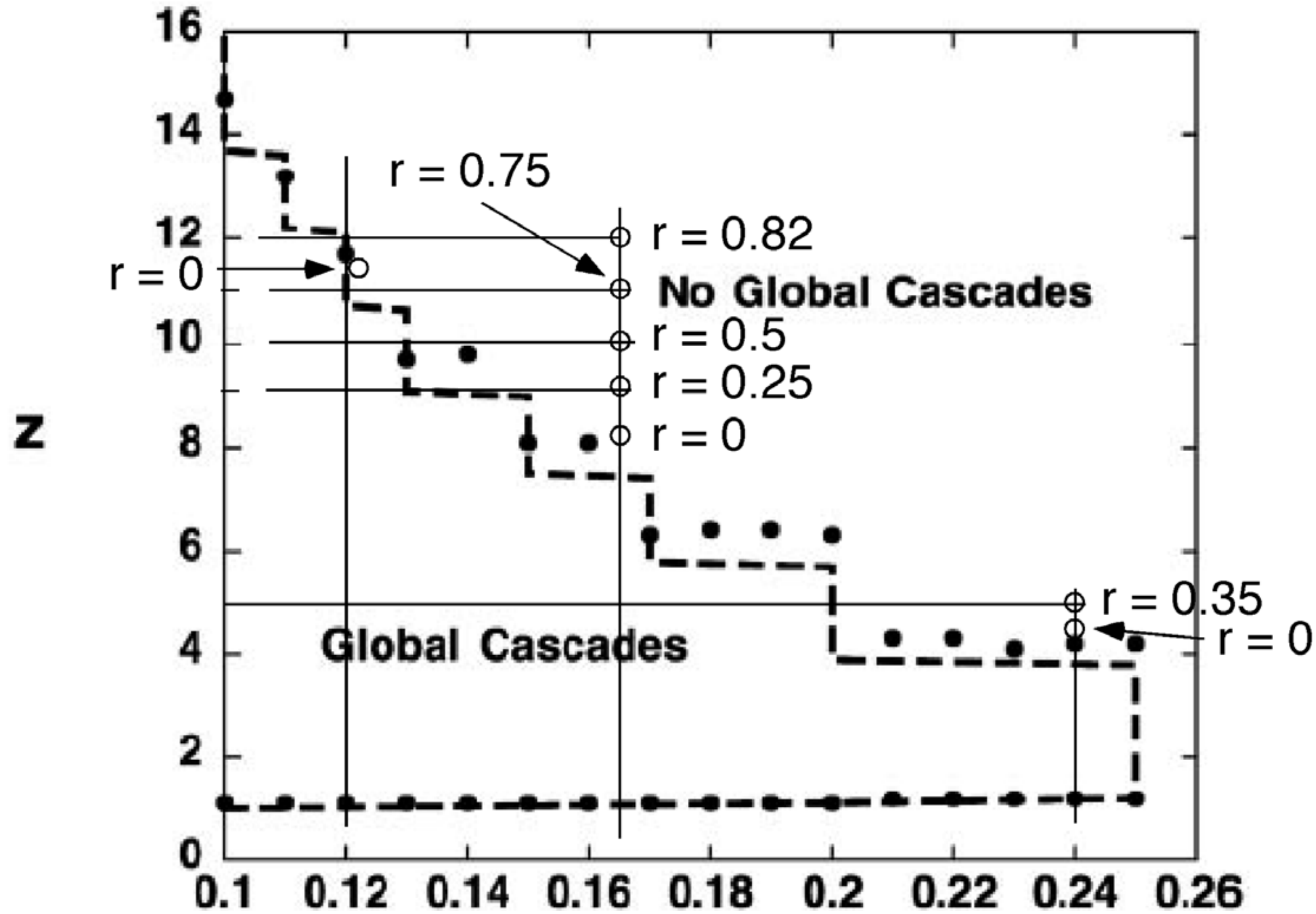
Theory and Simulations: Evolution of max One Short Failures (avg of 20 runs)





If one short exceeds the bound, a TNC almost always occurs.
If one short does not exceed the bound, a TNC almost never occurs.
Variation in one short can cause a TNC when mean is below bound.

When *r > 0* Cascades Occur for Bigger *z*



Increasing *r* generates larger vulnerable clusters

References

- [Callaway, Newman, Strogatz, Watts] "Network Robustness and Fragility: Percolation on Random Graphs," Phys Rev Lett (85) 25, Dec 18, 2000 pp 5468-5471
- [Dall'Asta] "Inhomogeneous Percolation Models for Spreading Phenomena in Random Graphs," arXiv:cond-mat/0509020 1 Sept 2005
- [Newman] "Structure and Function of Complex Networks," arXive:cond-mat/0303516 v1 25 March 2003 pp 20-23 and 38-39
- [Newman] "Random Graphs as Models of Networks" Chapter 2 in *Handbook of Graphs and Networks* ed. Bornholdt and Schuster, New York: Wiley 2005
- [Newman, Strogatz, Watts] "Random Graphs with Arbitrary Degree Distributions and Their Applications" arXive:cond-mat/0007235 v2 7 May 2001
- [Watts] "A Simple Model of Global Cascades on Random Networks" PNAS April 30, 2002, pp 5766-5771
- [Wikipedia] http://en.wikipedia.org/wiki/Percolation_theory and http://en.wikipedia.org/wiki/Percolation_threshold

More References

- D. E. Whitney, Dynamic theory of cascades on finite clustered random networks with a threshold rule, *Physical Review E*. E **82**, 066110 (2010)
- Granovetter, M., "Threshold Models of Collective Behavior," (1978) *Am J. Soc*. **83**, 1420-1443
- Lopez-Pintado, D., "Diffusion in Complex Social Networks," *Games and Economic Behavior*, forthcoming
- Morris, S. N., "Contagion," *Rev Econ Studies*, (67) 57-78 (2000)
- Rogers, E. M., *Diffusion of Innovations*, New York: Free Press, 2003
- Schelling, T.C., "A Study of Binary Choices with Externalities," *J. Conflict Resolution*, 17 (3) 381- 428 (1973)
- Valente, T. W. *Network Models of the Diffusion of Innovations*, Hampton Press, 1995

More References

- http://cnets.indiana.edu/tag/epidemic-modeling
- Multiscale mobility networks and the spatial spreading of infectious diseases. D.Balcan, V. Colizza, B. Gonsalves, H. Hu, J. J. Ramasco, A. Vespignani Proc Natl Acad Sci U S A **106**, 21484-21489 (2009).
- Predictability and epidemic pathways in global outbreaks of infectious diseases: the SARS case study V. Colizza, A. Barrat, M. Barthelemy, A. Vespignani BMC Medicine **5**, 34 (2007)
- Seasonal transmission potential and activity peaks of the new influenza A(H1N1): a Monte Carlo likelihood analysis based on human mobility D. Balcan, H. Hu, B. Goncalves, P. Bajardi, C. Poletto, J. J. Ramasco, D. Paolotti, N. Perra, M. Tizzoni, W. Van den Broeck, V. Colizza, A. Vespignani , BMC Medicine **7**, 45 (2009)

Backups

Generalized Random Networks

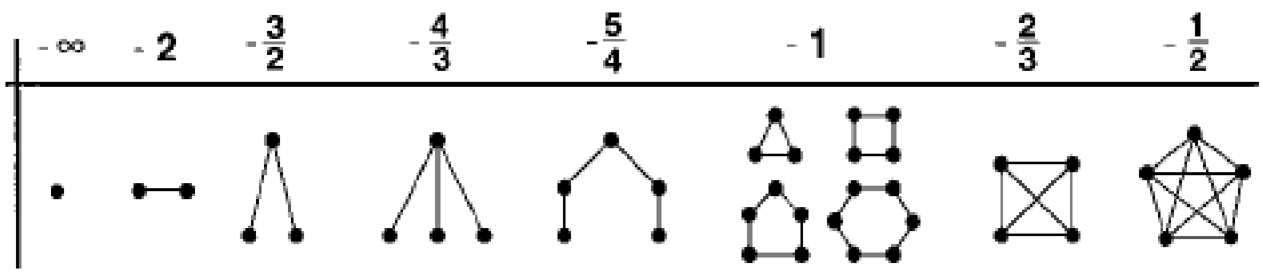
- E-R random network has a Poisson degree distribution
- Random networks can be built with arbitrary degree distributions, but software is required
- Newman says that it is better to generate a specific degree sequence from the distribution and then generate a network with that degree sequence in order to guarantee that the software uses the same degree sequence all the way through the generating process

Subgraph Shapes vs *p*

$$p \sim n^{a}, z \sim n^{a+1}$$

z	0	n^{-1}	$n^{-1/2}$	$n^{-1/3}$	$n^{-1/4}$	1	$n^{1/3}$	$n^{1/2}$
p	0	n^{-2}	$n^{-3/2}$	$n^{-4/3}$	$n^{-5/4}$	n^{-1}	$n^{-2/3}$	$n^{-1/2}$

a



z for n = 1000: 0 0.001 0.0316 0.1 0.178 1 10 31.6

The threshold probabilities at which different subgraphs appear in a random graph. For $pn^{3/2} \to 0$ the graph consists of isolated nodes. For $p \sim n^{-3/2}$ trees of order 3 appear, while for $p \sim n^{-4/3}$ trees of order 4 appear, but not many. At $p \sim n^{-1}$ trees of all orders are present and at the same time cycles of all orders appear, but again, not many. The probability $p \sim n^{-2/3}$ marks the appearance of complete subgraphs of order 4 and $p \sim n^{-1/2}$ corresponds to complete subgraphs or order 5. As *a* approaches 0 the graph contains complete subgraphs of increasing order.

R. Albert and A.-L. Barabási: Statistical mechanics of complex networks 56
Rev. Mod. Phys., Vol. 74, No. 1, January 2002

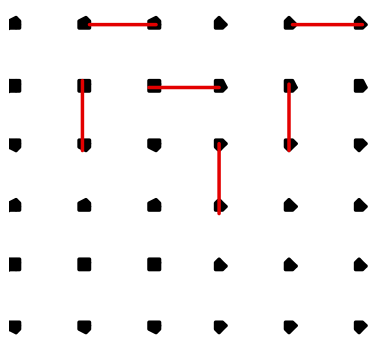
Site and Bond Percolation on Regular Graphs

- "The most common percolation model is to take a regular lattice, like a square lattice, and make it into a random network by randomly 'occupying' sites (vertices) or bonds (edges) with a statistically independent probability p. At a critical threshold p_c, long-range connectivity first appears, and this is called the **percolation threshold**." [see wikipedia reference "percolation threshold"]
- For a square grid, $p_c = 0.5$ for bond percolation and $p_c = 0.59274621$ for site percolation

Percolation Step by Step

Regular Networks: Occupancy

Each node has 2 neighbors. It must be linked to both for there to be a chance of a giant cluster. So $p_c = 1$.



Each node has 4 neighbors. It must be linked to at least 2 for there to be a chance of a giant cluster. So $p_c = 0.5$.

Random Networks: *z*

In a random network with n nodes each node has n neighbors. It must be linked to at least one for there to be a chance of a giant cluster. So $p_c = 1/n$. But $z = pn$ so this is the same as $z_c = 1$.

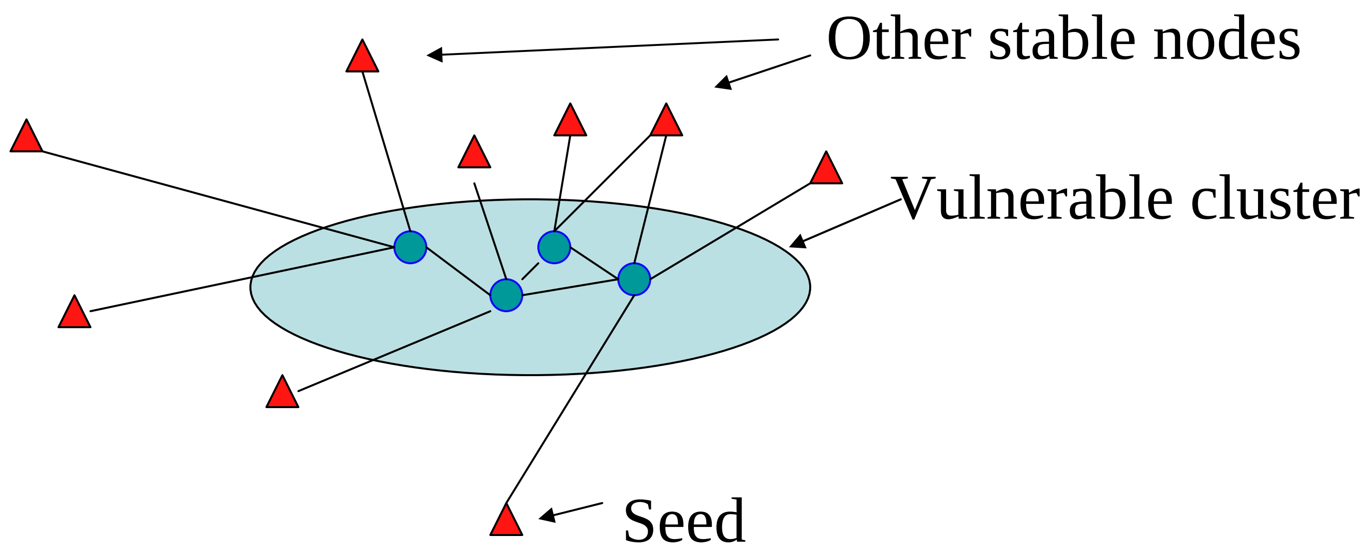
See Albert and Barabasi "Stat Mech of Complex Networks" for a detailed derivation

There is no proof or formula for p_c when d > 2 except for d > 19 and some special cases.

See Slade, Gordon, "Probabilistic Models of Critical Phenomena," *Princeton Companion to Mathematics*, edited by Timothy Gowers. Scheduled for publication in 2007. for a detailed discussion http://www.math.ubc.ca/~slade/

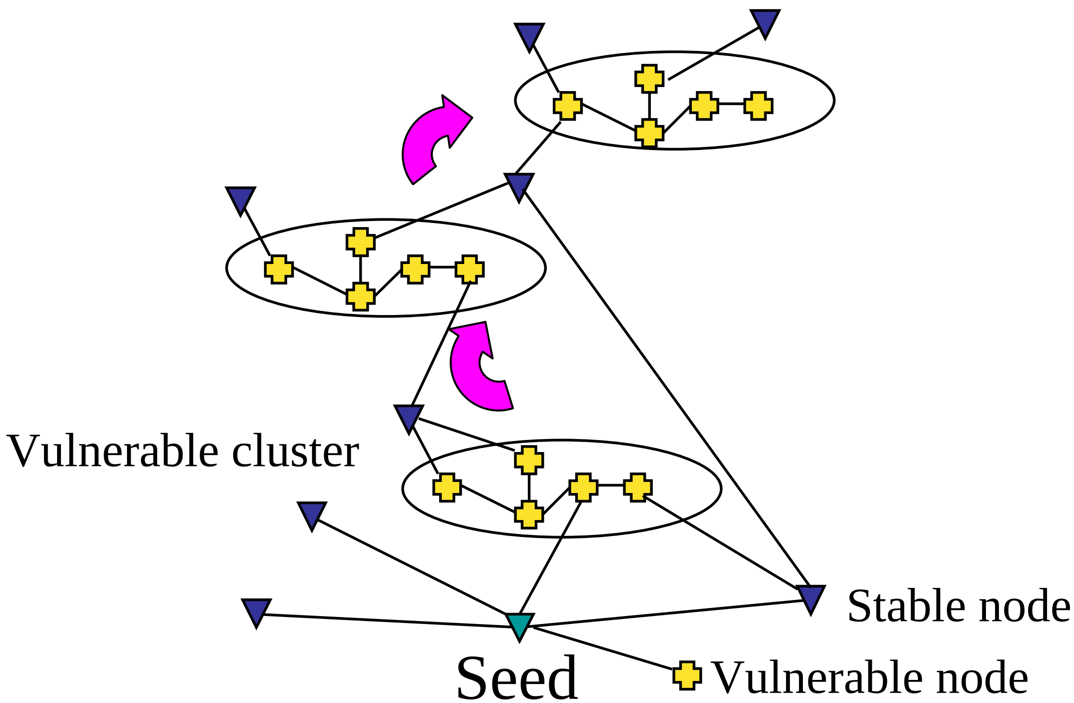
Vulnerable Clusters Multiply the Seed's Search Efficiency and Effectiveness

By itself, a single seed cannot flip a stable node



Vulnerable nodes have a few links to each other (average ~ 1.5) and more links to stable nodes outside their cluster. Working together, vulnerable nodes can flip stable nodes but most likely this happens only when vulnerable nodes co-exist in clusters

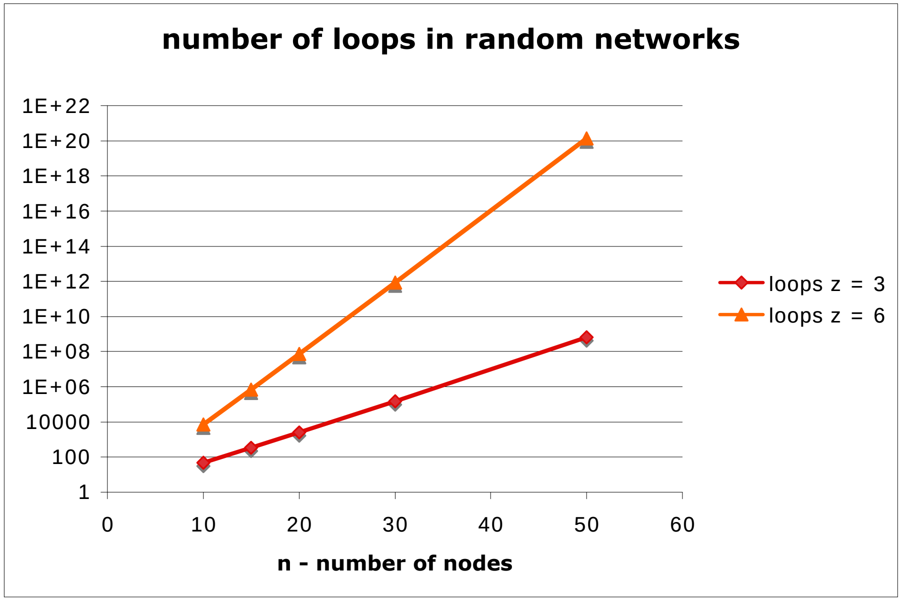
Cluster-Hopping Creates TNCs When Vulnerable Clusters Are Small

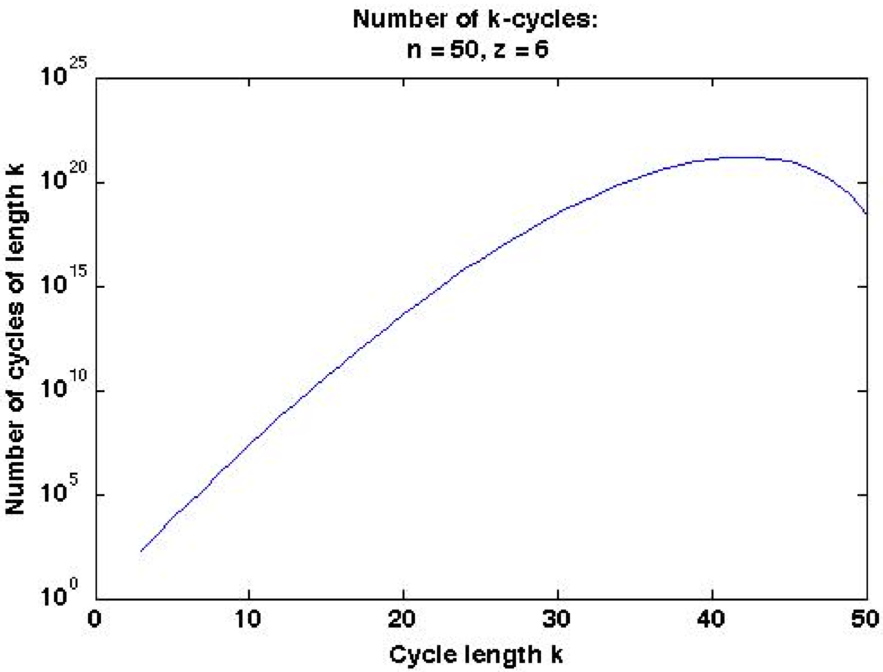


Whitney, ICCS, 2007

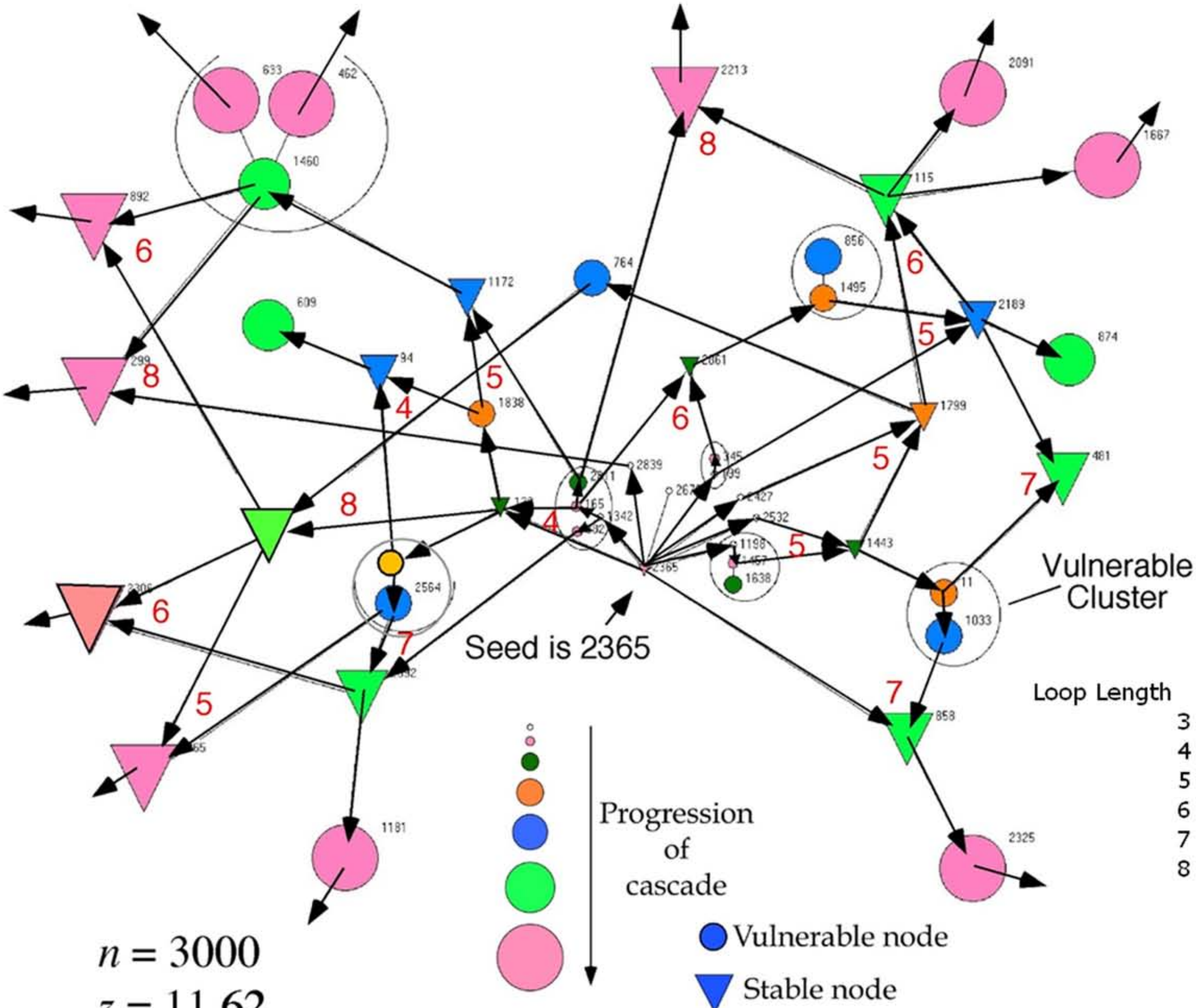
Do Random Networks Have Cycles?

$$Expected\ k-cycles = \frac{\binom{n}{k}k!}{2k}p^k \approx \frac{z^k}{2k} \text{ for } n >> k$$





Diestel, R., Graph Theory, 3rd edition online at <http://www.math.uni-hamburg.de/home/diestel/books/graph.theory/index.html> page 298



Loop Length	Number of Loops
3	261
4	2274
5	21114
6	204118
7	2028956
8	20581273

$n = 3000$

$z = 11.62$

Biggest cluster = 13

and cascades

Progression of a cascade.jpg

Percolation Theory for Random Graphs

- Most theory assumes we are dealing with a sparse network that has few or no closed loops, especially no small closed loops
- This is measured by the clustering coefficient, which is small for big random networks where the theory has been developed
- If there is no clustering or small closed loops then it is easy to calculate how many neighbors, second neighbors, third neighbors, etc, a given node has because no node is its own third neighbor and the probability that a node is its own n^{th} neighbor goes down as n goes up.
- If there are more n^{th} neighbors than $(n-1)^{th}$ neighbors for all n and the network is tree-like, then there is a giant cluster
- The calculations can be done for any random graph whose degree distribution is known, not just E-R random graphs, as long as there is negligible clustering

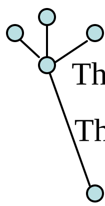
Variants of the Theory

- 1. The percolation (or cascade) proceeds when a link is established between two nodes. This is basic simple percolation described on the previous slide.
- 2. The percolation proceeds if a link is established with a node that is "vulnerable"
  - A) Vulnerability can be a function of *k* or it can be the same for all nodes (some number $0 \le b \le 1$)
  - B) Simple percolation has $b = 1$ for all nodes
  - C) Watts rumors cascade model has

$$b = 1 \text{ for } k \le K^*$$

$$b = 0 \text{ for } k > K^*$$

Derivation of Cascade Conditions (Newman)



Pick an edge leading from a node and follow it to a neighboring node.

What is the (excess) degree distribution of this neighbor?

If its degree = k, its edges are k-times more numerous than if its degree = 1 (think of the edge list)

But the fraction of nodes with degree k is p_k.

So the likelihood of encountering a node of degree k by this process is proportional to kp_k

$$\text{Distribution of excess degrees of neighbor} = q_{k-1} = \frac{kp_k}{\sum_{k=0}^{\infty} kp_k} \quad \text{or} \quad q_k = \frac{(k+1)p_{k+1}}{z}$$

$$\langle q \rangle = \text{avg excess degree of neighbor} = \sum_{k=0}^{\infty} kq_k = \frac{\sum_{k=0}^{\infty}(k+1)kp_{k+1}}{z} = \frac{\sum_{k=0}^{\infty}k(k-1)p_k}{z} = \frac{<k^2>-z}{z} = \frac{z_2}{z_1}$$

Ref: Newman, "Random Graphs as Models of Networks," Ch 2 in *Handbook of Graphs and Networks*, ed Bornholdt and Schuster

continuing

avg number of 2nd neighbors per 1st neighbor = $z_2 = < k^2 > - z$

avg number of 3rd neighbors per 2nd neighbor = $z_3 = z_2 = < k^2 > - z$

Avg number of *mth* neighbors = $z_1 \left[\frac{z_2}{z_1} \right]^{m-1}$

This diverges when $\left[\frac{z_2}{z_1} \right] = 1$

(from prev slide)

$$\frac{\sum_{k=0}^{\infty} k(k-1) p_k}{z} = 1$$

Generalizeable
Cascade condition

or

$$\sum_{k=0}^{\infty} k(k-1) p_k = z$$

Molloy-Reed criterion

For E - R $\langle k^2 \rangle = \langle k \rangle^2 = z^2$

So, for E - R this is the same as $z = 1$

(See notes)

continuing

For the case where all nodes have vulnerability = b:

$$b\sum_{k=0}^{\infty} k(k-1)p_k = z$$

See notes

For the case where vulnerability is a function ρ_k of k

$$\sum_{k=0}^{\infty} k(k-1)\rho_k p_k = z$$

Watts rumor cascade model:

$$\rho_k = \begin{cases} 1 \text{ for } k \le K^* \\ 0 \text{ for } k > K^* \end{cases}$$

Supporting derivations of these typically use generating functions

Rules for Simulating Cascades

- Build a random network with some value of z and set the value of K*
- Choose a node at random (the seed) and flip it
- Find its neighbors and flip all that are vulnerable
- Find their neighbors and flip all that can be flipped
  - "Vulnerable" ones flip if one neighbor flipped
  - "First-order" stable ones flip if two neighbors flipped, etc
- Keep going until all nodes have flipped that can
- Use some criterion to say if a global cascade has happened or not
- Watts made a new network each time but I reused the network to save time. This permitted me to examine it.

ESD.342 Network Representations of Complex Engineering Systems
Spring 2010