Fuzzy and Rough Sets
Part II

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Overview

- Fuzzy sets
- Fuzzy logic and rules
- Rough sets and rules
- An example of a method for mining rough/fuzzy rules
- Uncertainty revisited

Crisp Sets

- A set with a characteristic function is called *crisp*
- Crisp sets are used to formally characterize a *concept*, e.g., even numbers
- Crisp sets have clear cut boundaries, hence do not reflect uncertainty about membership

Fuzzy Sets

- Zadeh (1965) introduced "Fuzzy Sets" where he replaced the characteristic function with membership
- $\chi_S: U \rightarrow \{0,1\}$ is replaced by
  $m_S : U \rightarrow [0,1]$
- Membership is a generalization of characteristic function and gives a "degree of membership"
- Successful applications in control theoretic settings (appliances, gearbox)

Fuzzy Sets

- Example: Let S be the set of people of normal height
- Normality is clearly not a crisp concept

Crisp Characterizations of Fuzzy Sets

- Support in U

  $\text{Support}_U(S) = \{x \in U \mid m_S(x) > 0\}$

- Containment

  $A \subseteq B$ if and only if

  $m_A(x) \leq m_B(x)$ for all $x \in U$

- There are non-crisp versions of the above

Fuzzy Set Operations

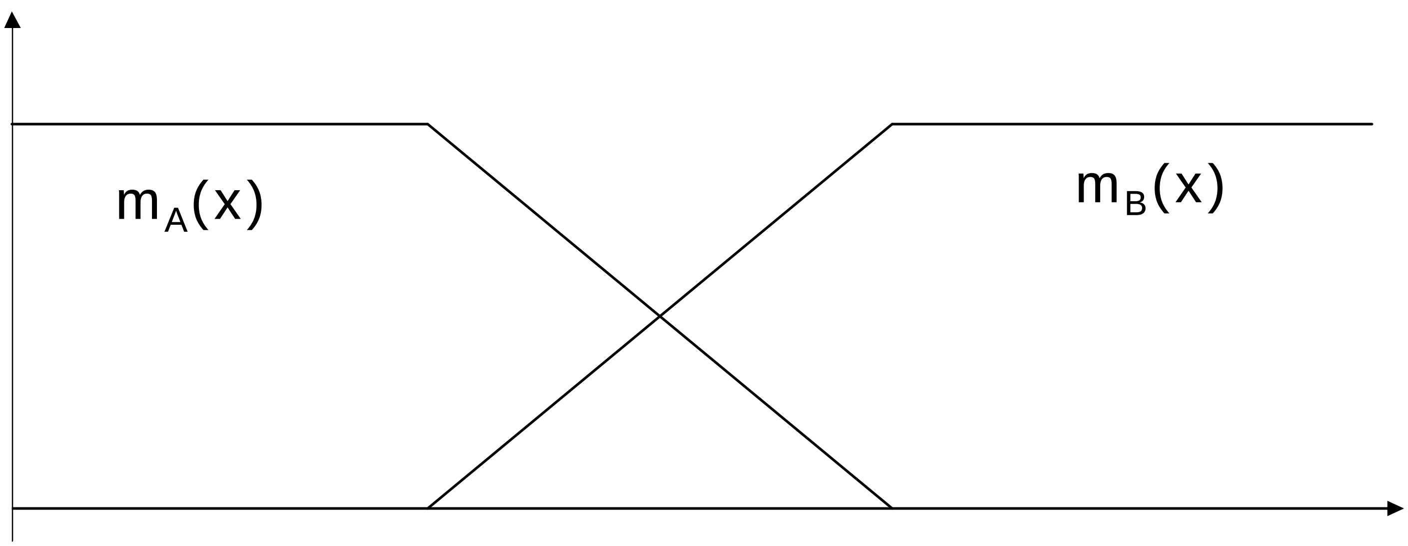
- Union

  $m_{A \cup B}(x) = max(m_A(x), m_B(x))$
- Intersection

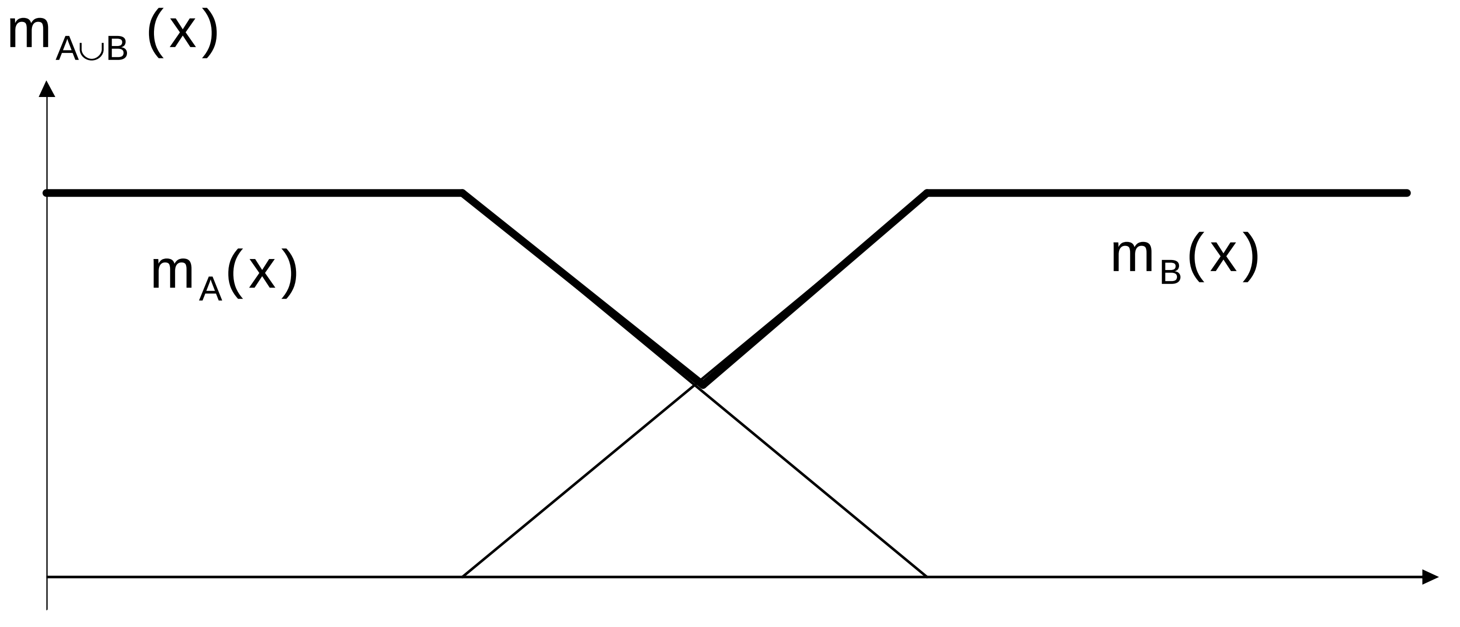
  $m_{A \cap B}(x) = min(m_A(x), m_B(x))$
- Complementation

  $m_{U-A}(x) = 1 - m_A(x)$
- Note that other definitions exist too

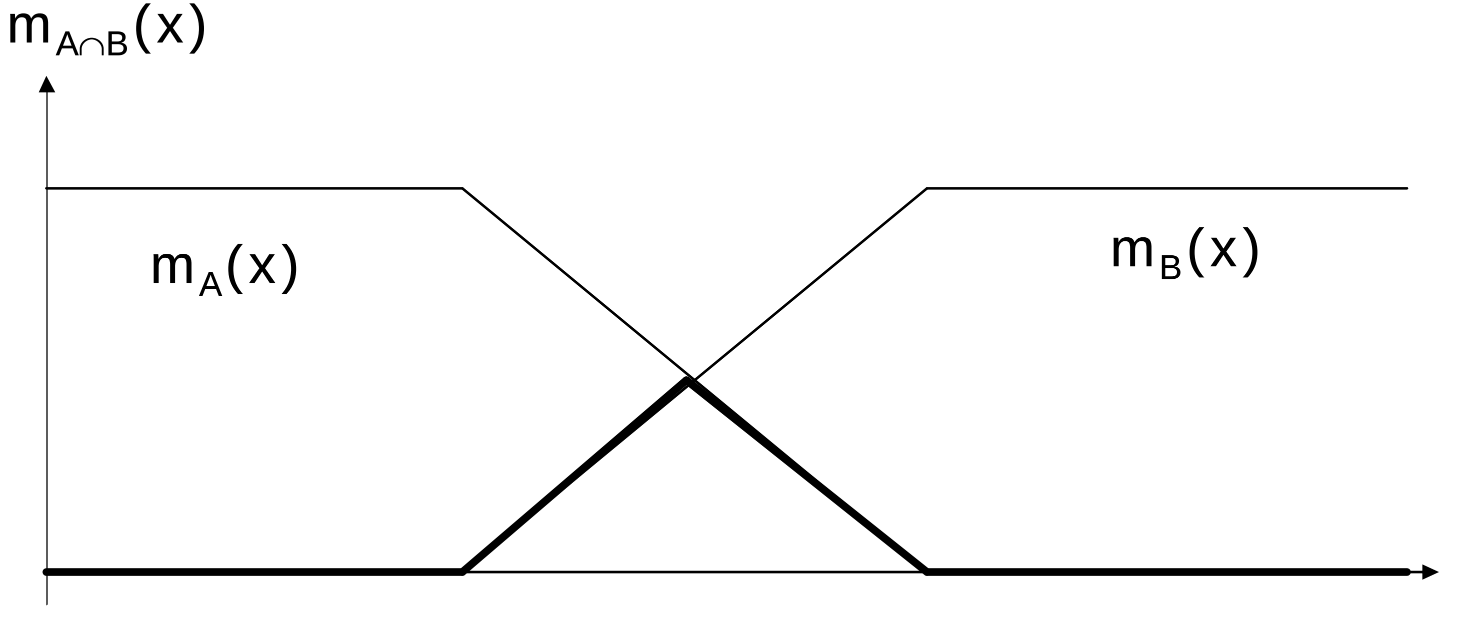
Fuzzy Memberships Example



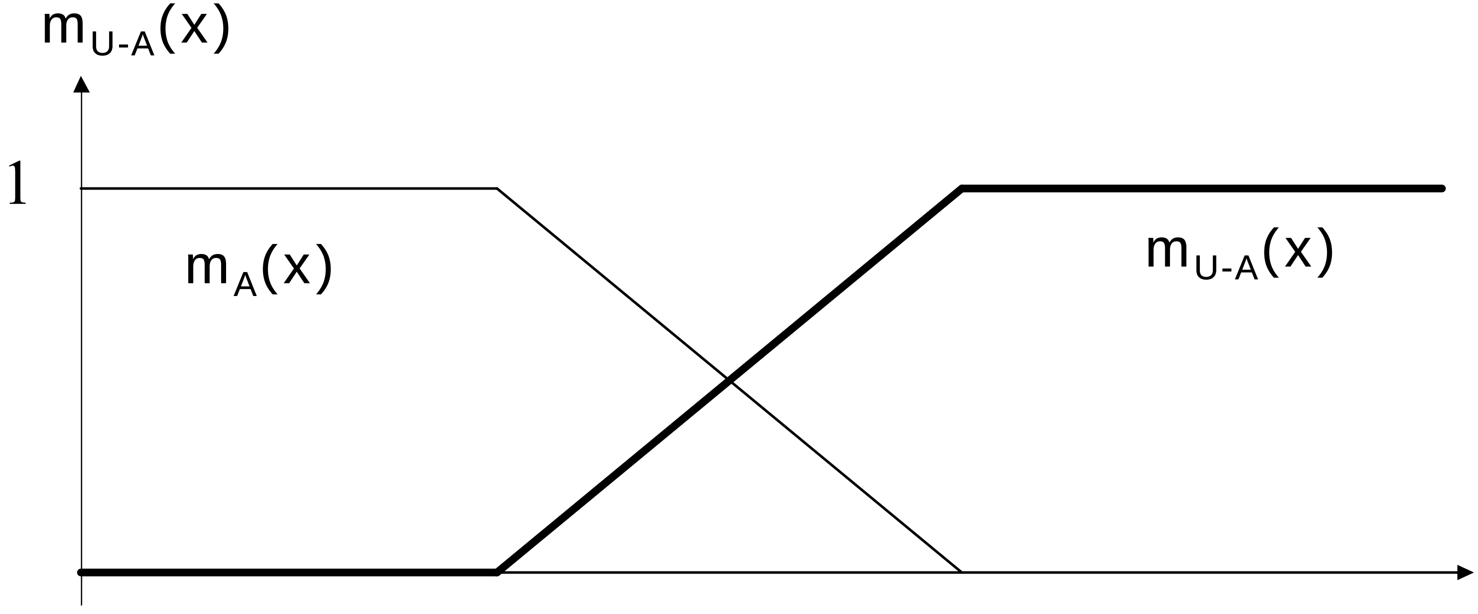
Fuzzy Union Example



Fuzzy Intersection Example



Fuzzy Complementation Example



Fuzzy Relations

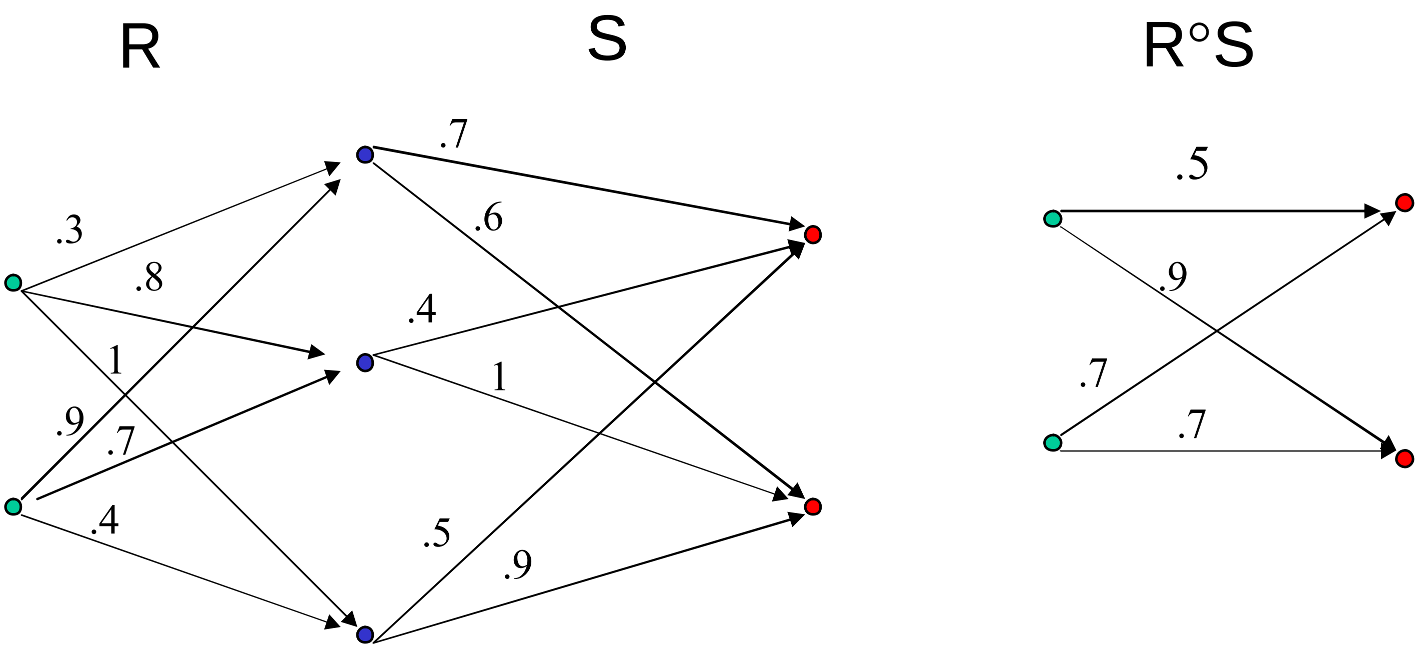
- The fuzzy relation R between Sets X and Y is a fuzzy set in the Cartesian product $X \times Y$
- m_R: $X \times Y \rightarrow [0,1]$ gives the degree to which x and y are related to each other in R.

Composition of Relations

- Two fuzzy relations R in X×Y and S in Y×Z can be composed into R∘S in X×Z as

$$m_{R \circ S}(x,z) = \max_{y \in Y}[\min[m_R(x,y),\ m_S(y,z)]]$$

Composition Example



Probabilities of Fuzzy Events

- "Probability of cold weather tomorrow"
- $U = \{x_1, x_2, ..., x_n\}$, p is a probability density, A is a fuzzy set (event) in U

$$P(A) = \sum_{i=1}^{n} m_A(x_i)\, p(x_i)$$

Defuzzyfication

- Finding a single representative for a fuzzy set A in U = $\{x_i | i \text{ in } \{1,\ldots n\}\}$
- Max: x in U such that $m_A(x)$ is maximal
- Center of gravity:

$$\frac{\sum_{i=1}^{n} x_i m_A(x_i)}{\sum_{i=1}^{n} m_A(x_i)}$$

Alpha Cuts

- A is a fuzzy set in U
- $A_\alpha = \{x \mid m_A(x) \geq \alpha\}$ is the α-cut of A in U
- *Strong* α-cut is

  $A_\alpha = \{x \mid m_A(x) > \alpha\}$
- Alpha cuts are crisp sets

Fuzzy Logic

- Different views
  - Foundation for reasoning based on uncertain statements
  - Foundation for reasoning based on uncertain statements where fuzzy set theoretic tools are used (original Zadeh)
  - As a multivalued logic with operations chosen in a special way that has some fuzzy interpretation

Fuzzy Logic

- Generalization of proposition over a set
- Let $\chi_S : U \rightarrow \{0,1\}$ denote the characteristic function of the set S
- Recall that in "crisp" logic

  $I(p(x)) = p(x) = \chi_{T(p)}(x)$

  where p is a proposition and T(p) is the corresponding truth set

Fuzzy Logic

- We extend the proposition

  p:U ® {0,1}

  to be a fuzzy membership

  p:U ® [0,1]

- The fuzzy set associated with p corresponds to the truth set T(p) and p(x) is the degree of truth of p for x
- We extend the interpretation of logical formulae analogously to the crisp case

Fuzzy Logic Sematics

- Basic operations:
  - $I(p(x)) = p(x)$
  - $I(\alpha \vee \beta) = \max(I(\alpha), I(\beta))$
  - $I(\alpha \wedge \beta) = \min(I(\alpha), I(\beta))$
  - $I(\sim\alpha) = 1 - I(\alpha)$

Fuzzy Logic Sematics

- Implication:
  - Kleene-Dienes

    I(α ® β) = max(I(~α),I(β))
- Dubois and Prade (1992) analyze other definitions of Implications
  - Zadeh

    I(α ® β) = max(I(~α),min(I(α), I(β)))

Fuzzy Rules

- "If x in A then y in B" is a relation R between A and B
- Two model types
  - Implicative: (x in A ® y in B) is an upper bound
  - Conjunctive: (x in A ^ y in B) is a lower bound
  - Crisp motivation:

$$\chi_A(x) \wedge \chi_B(y) \le \chi_R(x,y) \le (1 - \chi_A(x)) \vee \chi_B(y)$$

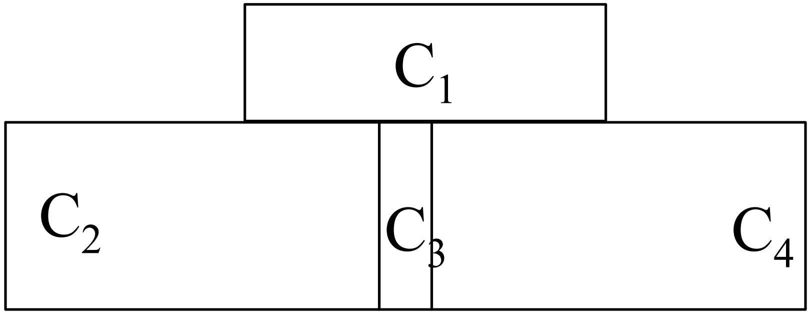
Conjunctive Rule application

- R:U×U ® [0,1] is a rule
  If p(x) then q(y)
- Using a generalized Modus Ponens
  A’
  A ® B
  B’
  ———————————
  we get that
  B’ = A’ ° R
  $B'(y) = \max_x[\min[A'(x),R(x,y)]]$

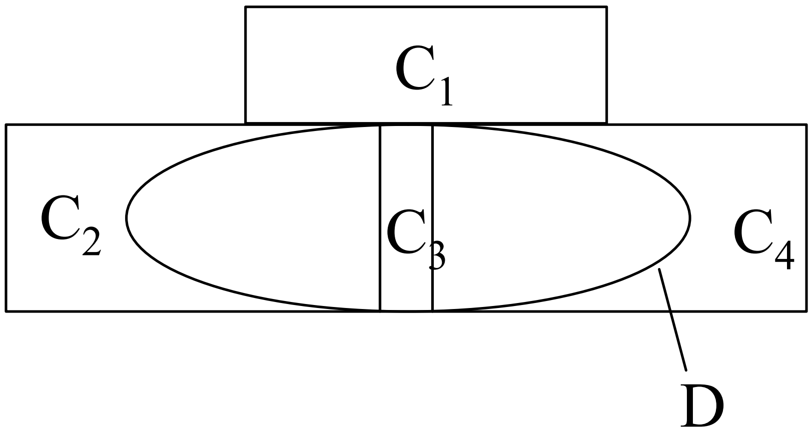
Rough Sets

- Pawlak 1982
- Approximation of sets using a collection of sets.
- Related to fuzzy sets (Zadeh 1965), in that both can be viewed as representations of uncertainty regarding set membership.

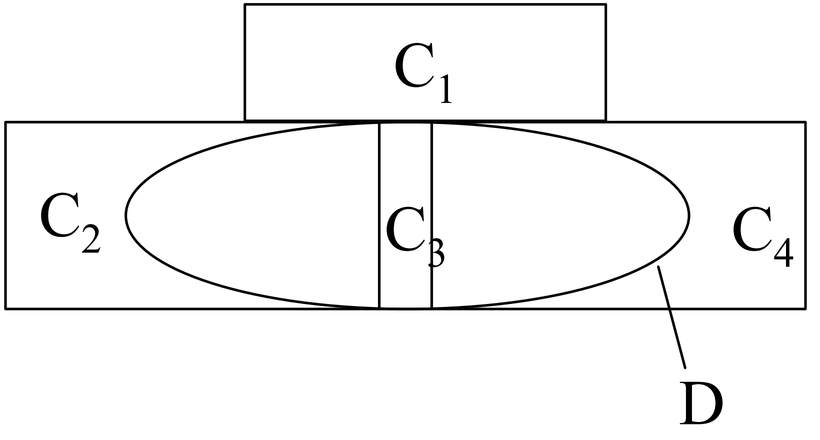
Rough Set: Set Approximation



Rough Set: Set Approximation



Rough Set: Set Approximation



- Approximation of D by $\{c_1, c_2, c_3, c_4\}$:
  - c_1 definitely outside
  - c_3 definitely inside: lower approximation
  - $c_2 \cup c_4$ are boundary
  - $c_2 \cup c_3 \cup c_4$ are upper approximation

Rough Set: Set Approximation

- Given a collection of sets *C={c₁, c₂, c₃,…}* and a set D, we define:
  - *Lower approximation* of D by C,

$$D^L = \bigcup C_i \text{ such that } C_i \cap D = C_i$$

  - *Upper approximation* of D by C,

$$D^U = \bigcup C_i \text{ such that } C_i \cap D \neq \emptyset$$

  - *Boundary* of D by C,

$$D_L^U = D^U - D_L$$

Rough Set: Definition

- A set D is *rough* with respect to a collection of sets C if it has a non-empty boundary when approximated by C. Otherwise it is *crisp*.

Rough Set: Information System

- Universe U of elements, e.g., patients.
- Set *A* of features (attributes), functions *f* from U to some set of values V_f.
- (U,A) – *information system*

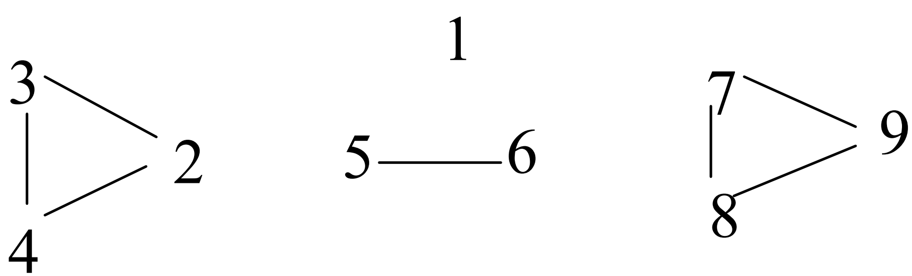
Object no.	a	b	c	d
1	0	0	1	0
2	0	1	1	1
3	0	1	1	0
4	0	1	1	0
5	1	0	0	1
6	1	0	0	1
7	1	1	0	1
8	1	1	0	1
9	1	1	0	0

$$U = \{1,2,3,4,5,6,7,8,9\}$$
$$A = \{a,b,c,d\}$$
$$V_a = V_b = V_c = V_d = \{0,1\}$$

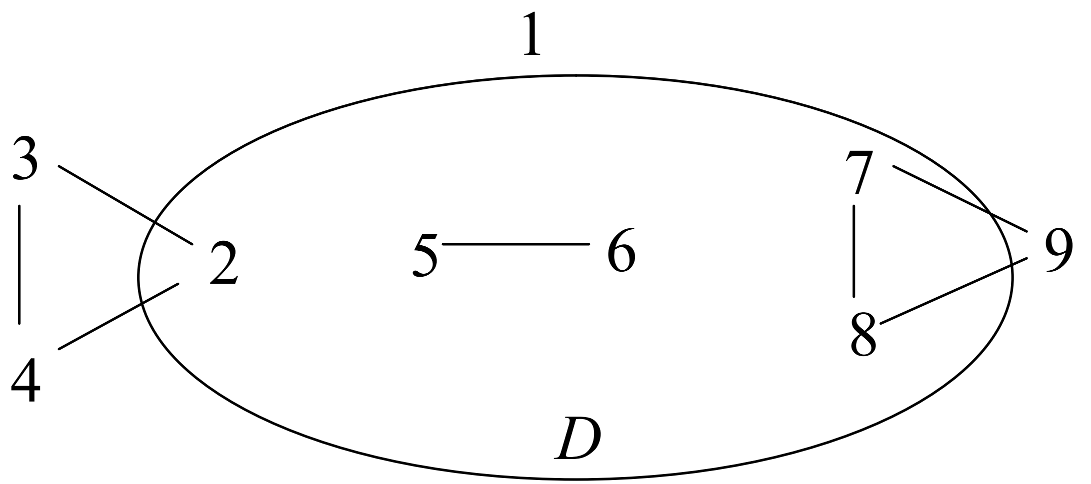
Rough Sets: Partition of U

- $E = \{(i,j) \in U \times U \mid abc(i) = abc(j)\}$, equivalence relation on U
- $E(1) = \{1\} = C_1$
- $E(2) = E(3) = E(4) = \{2,3,4\} = C_2$
- $E(5) = E(6) = \{5,6\} = C_3$
- $E(7) = E(8) = E(9) = \{7,8,9\} = C_4$

Object no.	abc	d
1	(0,0,1)	0
2	(0,1,1)	1
3	(0,1,1)	0
4	(0,1,1)	0
5	(1,0,0)	1
6	(1,0,0)	1
7	(1,1,0)	1
8	(1,1,0)	1
9	(1,1,0)	0



Rough Sets: Approximating D



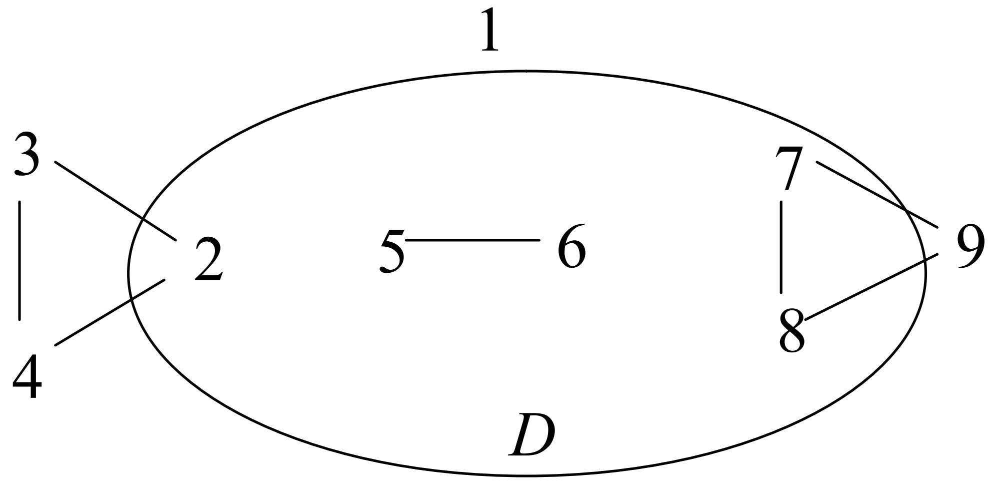
$$D^U = \{2,3,4,5,6,7,8,9\} = C_2 \cup C_3 \cup C_4$$

$$D_L = \{5,6\} = C_3$$

$$D^U - D_L = \{2,3,4,7,8,9\} = C_2 \cup C_4$$

Rough Sets: Approximate membership *d*

$$d(i) = \frac{|D \cap E(i)|}{|E(i)|}$$



- *d(1) = 0*
- *d(2) = d(3) = d(4) = 1/3*
- *d(5) = d(6) = 1*
- *d(7) = d(8) = d(9) = 2/3*

Rough Sets: Data Compression

Information: Partition given by equivalence.

Find minimal sets of features that preserve information in table.

Object no.	a	b	c	d
1	0	0	1	0
2	0	1	1	1
3	0	1	1	0
4	0	1	1	0
5	1	0	0	1
6	1	0	0	1
7	1	1	0	1
8	1	1	0	1
9	1	1	0	0

Object no.	a	b	d
1	0	0	0
2	0	1	1
3	0	1	0
4	0	1	0
5	1	0	1
6	1	0	1
7	1	1	1
8	1	1	1
9	1	1	0

Rough Sets: Discernibility Matrix

- $M_A = \{m_{ij}\}, A = \{a,b,c\}$
- m_{ij} = *{a Î A | a(k) ¹ a(l), k Î* $C_{i,}$ *l Î* C_j *}*

$M_A =$

{}	{b}	{a,c}	{a,b,c}
{b}	{}	{a,b,c}	{a,c}
{a,c}	{a,b,c}	{}	{b}
{a,b,c}	{a,c}	{b}	{}

Object no.	a	b	c
1	0	0	1
2,3,4	0	1	1
5,6	1	0	0
7,8,9	1	1	0

C = { {b},{a,c}{a,b,c} } – set of non-empty entries of M_A
Minimal sets that have non-empty intersection with all elements of *C* are *{a,b}* and *{b,c}* (Finding: Combinatorial)
These are called reducts of (U,A)
A reduct is a minimal set of features that preserves the partition.

Rough Sets: Extending *d*

- Problem: we only have the *d* value for 4 of 8 possible input values. What is *d(1,1,1)* ?
- By using compressed data that preserves the partition, we cover more of the feature space. All of it in this case. *d(1,1,1)* = *d(1,1)* = *2/3.*

abc	d
(0,0,1)	0
(0,1,1)	1/3
(1,0,0)	1
(1,1,0)	2/3

ab	d
(0,0)	0
(0,1)	1/3
(1,0)	1
(1,1)	2/3

Rough Sets: Extending *d*

- Problem: extension not unique (and can extend to different parts of feature space).
- $d(1,1,1) = d(1,1) = 1/3$.
- Possible solution: generate several extensions and combine by voting. Generating all extensions is combinatorial.
- $d(1,1,1) = (2/3 + 1/3)/2 = 1/2$

abc	d
(0,0,1)	0
(0,1,1)	1/3
(1,0,0)	1
(1,1,0)	2/3

bc	d
(0,0)	1
(0,1)	0
(1,0)	2/3
(1,1)	1/3

Rough Sets: Classification rules

Object no.	a	b	c	d
1	0	0	1	0
2	0	1	1	1
3	0	1	1	0
4	0	1	1	0
5	1	0	0	1
6	1	0	0	1
7	1	1	0	1
8	1	1	0	1
9	1	1	0	0

ab	d
(0,0)	0
(0,1)	1/3
(1,0)	1
(1,1)	2/3

Rules with right hand side support numbers:

a(0) AND b(0) => d(0) (1)

a(0) AND b(1) => d(1) OR d(0) (1, 2)

a(1) AND b(0) => d(1) (2)

a(1) AND b(1) => d(1) OR d(0) (2, 1)

A Proposal for Mining Fuzzy Rules

- Recipe:
  1. Create rough information system by fuzzy discretization of data
  2. Compute rough decision rules
  3. Interpret rules as fuzzy rules

Fuzzy Discretization

- $A_1, A_2, \ldots, A_n$ are fuzzy sets in U
- disc: $U \rightarrow \{1,2,\ldots,n\}$

  $\text{disc}(x) = \{i \mid m_{A_i}(x) = \max\{m_{A_j}(x) \mid j \in \{1,2,\ldots,n\}\}$
- disc selects the index of the fuzzy set that yields the maximal membership
- Information system: subject each attribute value to disc

Fuzzy Rough Rules: Example

$A_1(3.14) = 0.6$
$A_1(0.1) = 0.3$
$A_2(3.14) = 0.5$
$A_2(0.1) = 0.8$

Object no.	a	d
1	3.14	0
2	0.1	1

Object no.	a	d
1	1	0
2	2	1

if A1 then d=0
if A2 then d=1

Uncertainty

- Fuzzy sets can be said to model inherent vagueness

  Bob is "tall" - vagueness in the meaning of "tall", not in Bob's height

- Rough sets can be said to model ambiguity due to lack of information

And…

- Thank you for your attention