# Logic I - Session 23

Completeness of PD

# Soundness recap

- Last time we sketched a proof that PD is sound.
  - I.e., if $\Gamma \vdash \mathbf{P}$ in PD then $\Gamma \vDash \mathbf{P}$.
- The main part of the proof is proving that for any derivation:

  If $\Gamma i \vDash \mathbf{P}i$ for all $i \leq k$, then $\Gamma k{+}1 \vDash \mathbf{P}k{+}1$.

  - We prove this by showing that it holds for each rule that could justify line k+1.

# Soundness recap

- The strategy for the individual cases goes like this:
  - Given the rule justifying line k+1, try to draw conclusions about the form of **P**k+1.
  - Then draw conclusions about the structure of the derivation above line k+1 and about the forms of sentences on earlier lines, e.g. **Q**i and **R**j.
  - Apply the inductive hypothesis to **Q**i and **R**j.
  - Note the relationships among Γi, Γj, and Γk+1.
  - Draw conclusions about relationship between Γk+1 and **Q**i and **R**j.
  - Put this together with semantic definitions and

# Completeness

- Next up: prove if $\Gamma \vDash \mathbf{P}$ then $\Gamma \vdash \mathbf{P}$ in PD
- Remember the main strategy for completeness of SD.
- We argued that for sets of SL sentences:
  - Any C-SD set is a subset of a MC-SD set
  - Every MC-SD set is TF-C
  - Every subset of a TF-C set is TF-C.
  - So any C-SD set is TF-C.
- We then appealed to connections between consistency and entailment and derivability.
- We'll have a similar strategy for PD.

# Preliminary definitions

- MC-PD: $\Gamma^*$ is Maximally Consistent in PD iff $\Gamma^*$ is consistent in PD and $\Gamma^* \cup \{\mathbf{P}\}$ is inconsistent for any **P** not already in $\Gamma^*$.
- ∃C: Γ is Existentially Complete iff for each sent. in Γ of the form **(∃x)P**, there's a substitution instance of **(∃x)P** in Γ, e.g. **P(a/x)**.
- ES sets: Set $\Gamma_e$ is Evenly Subscripted iff it is the result of doubling the subscript of every i.c. in Γ.

# Completeness

$\Gamma \nvdash$ **P**

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ES-variant of $\Gamma \cup$ {~**P**} is C-PD

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ES-variant of $\Gamma \cup$ {~**P**} $\subseteq$ a MC-∃C-PD set $\Gamma^*$ (11.4.4)

↓

If $\Gamma^*$ is MC-∃C-PD then $\Gamma^*$ is Q-C (11.4.8)

↓

$\Gamma \cup$ {~**P**} $\subseteq$ a Q-C set $\Gamma^*$

↓

$\Gamma \cup$ {~**P**} is Q-C

↓

$\Gamma \nvDash$ **P**

$\Gamma \nvdash$ **P**

↓

The ES-variant of $\Gamma \cup$ {~**P**} is C-PD

- Assume $\Gamma \nvdash$ **P**.
  - Then if $\Gamma \cup$ {~**P**} were IC-PD, then we could derive some **Q** and ~**Q** from $\Gamma \cup$ {~**P**}.
    - And in that case, from $\Gamma$ we could derive **P** by ~E, contradiction the assumption that $\Gamma \nvdash$ **P**.
- So $\Gamma \cup$ {~**P**} is C-PD.
- Now we want to show that the ES variant of $\Gamma \cup$ {~**P**} is C-PD.
- So lets show that for any $\Gamma$, if $\Gamma$ is C-PD, then $\Gamma$e is C-PD.

$\Gamma \nvdash$ **P**

↓

The ES-variant of $\Gamma \cup \{\sim$**P**$\}$ is C-PD

- Suppose Γ is C-PD. Then Γe is the result of doubling the subscript on each individual constant in each sentence in Γ.
- To show that Γe is C-PD:
  - Suppose, for reductio, that Γe were IC-PD.
  - Then we could derive some **Q** and ~**Q** from Γe.
  - And then a certain **Q*** and ~**Q*** would be derivable from Γ, contradicting our assumption that Γ is C-PD.
  - Let's show how this reductio goes.

$\Gamma \nvdash \mathbf{P}$

↓

The ES-variant of $\Gamma \cup \{\sim\mathbf{P}\}$ is C-PD

- Suppose, for reductio, that Γe is IC-PD.
- Then there's a derivation from members of Γe to **Q** and ~**Q**.
- Halve the subscripts on each i.c. in the derivation, and you'll end up with premises that will all be members of Γ.
- The new sequence will be a derivation showing that Γ is IC-PD.
  - (There's a minor complication here that we'll skip...)
- This contradicts our assumption that Γ is C-PD.
- So Γe is C-PD.
- So for any Γ, if Γ is C-PD, then Γe is C-PD.
- So since $\Gamma \cup \{\sim\mathbf{P}\}$ is C-PD, an ES-variant of it is C-PD.

# Completeness

$\Gamma \nvdash \mathbf{P}$

↓ ☑

An ES-variant of $\Gamma \cup \{\sim\mathbf{P}\}$ is C-PD

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Any ES-variant of $\Gamma \cup \{\sim\mathbf{P}\} \subseteq$ a MC-∃C-PD set $\Gamma^*$

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If $\Gamma^*$ is MC-∃C-PD then $\Gamma^*$ is Q-C (11.4.8)

↓

$\Gamma \cup \{\sim\mathbf{P}\} \subseteq$ a Q-C set $\Gamma^*$

↓

$\Gamma \cup \{\sim\mathbf{P}\}$ is Q-C

↓

$\Gamma \nvDash \mathbf{P}$

$\Gamma \cup \{\sim\mathbf{P}\}$ is Q-C

↓

$\Gamma \nvDash \mathbf{P}$

- Suppose $\Gamma\cup\{\sim\mathbf{P}\}$ is Q-C.
  - Then there's an interpretation I′ that mem Γ and ~**P** true.
  - Now suppose for reductio that $\Gamma \vDash \mathbf{P}$.
    - Then every I that mem Γ true makes **P** true.
    - So I′ would have to make **P** and ~**P** true, which is impossible given the def. of ~
  - So $\Gamma \nvDash \mathbf{P}$.
- So if $\Gamma\cup\{\sim\mathbf{P}\}$ is Q-C, then $\Gamma \nvDash \mathbf{P}$.

# Completeness

$\Gamma \nvdash \mathbf{P}$

↓ ☑

An ES-variant of $\Gamma \cup \{\sim\mathbf{P}\}$ is C-PD

↓

Any ES-variant of $\Gamma \cup \{\sim\mathbf{P}\} \subseteq$ a MC-∃C-PD set $\Gamma^*$

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If $\Gamma^*$ is MC-∃C-PD then $\Gamma^*$ is Q-C (11.4.8)

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$\Gamma \cup \{\sim\mathbf{P}\} \subseteq$ a Q-C set $\Gamma^*$

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$\Gamma \cup \{\sim\mathbf{P}\}$ is Q-C

↓ ☑

$\Gamma \nvDash \mathbf{P}$

$\Gamma \cup \{\sim\mathbf{P}\} \subseteq$ a Q-C set $\Gamma^*$

↓

$\Gamma \cup \{\sim\mathbf{P}\}$ is Q-C

- Suppose $\Gamma \cup \{\sim\mathbf{P}\}$ is a subset of a Q-C set $\Gamma^*$.
- This means every member of $\Gamma \cup \{\sim\mathbf{P}\}$ is a member of $\Gamma^*$.
- Since $\Gamma^*$ is Q-C, there's an interpretation I′ that mem $\Gamma^*$ true.
- If every member of $\Gamma^*$ is true on I′, then since every member of $\Gamma \cup \{\sim\mathbf{P}\}$ is a member of $\Gamma^*$, every member of $\Gamma \cup \{\sim\mathbf{P}\}$ is true on I′.
- So every member of $\Gamma \cup \{\sim\mathbf{P}\}$ is true on I′.
- So there's an interpretation that mem $\Gamma \cup \{\sim\mathbf{P}\}$ true.
- So by def., $\Gamma \cup \{\sim\mathbf{P}\}$ is Q-C.

# Completeness

$\Gamma \nvdash$ **P**

↓ ☑

An ES-variant of $\Gamma \cup \{\sim$**P**$\}$ is C-PD

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Any ES-variant of $\Gamma \cup \{\sim$**P**$\}$ $\subseteq$ a MC-∃C-PD set $\Gamma^*$

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If $\Gamma^*$ is MC-∃C-PD then $\Gamma^*$ is Q-C (11.4.8)

↓

$\Gamma \cup \{\sim$**P**$\}$ $\subseteq$ a Q-C set

↓ ☑

$\Gamma \cup \{\sim$**P**$\}$ is Q-C

↓ ☑

$\Gamma \nvDash$ **P**

$\Gamma^*$ is Q-C (11.4.8)

↓

$\Gamma \cup \{\sim\mathbf{P}\} \subseteq$ a Q-C set $\Gamma^*$

- We know that the ES-variant of $\Gamma \cup \{\sim\mathbf{P}\}$ is a subset of a Q-C set, since the ES-variant is $\subseteq \Gamma^*$ and $\Gamma^*$ is Q-C.
- We'll show that for any Γ, if the ES-variant Γe is Q-C, Γ is Q-C.
- Since Γe is Q-C, some interpretation Ie mem Γe true.
- Let I be just like Ie except that where Ie assigns objects to constants with even subscripts, I assigns the same objects to corresponding constants with their subscripts halved.
  - E.g., if Ie(**a2**)=John, Ie(**a86**)=Bill, Ie(**c12**)=France,
    I(**a1**)=John, I(**a43**)=Bill, I(**c6**)=France
- Then each subscript difference b/t Γe and Γ is compensated for by a subscript difference between Ie and I.
  - So I says the same things about the same objects as Ie.

# Completeness

$\Gamma \nvdash \mathbf{P}$

↓ ☑

An ES-variant of $\Gamma \cup \{\sim\mathbf{P}\}$ is C-PD

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Any ES-variant of $\Gamma \cup \{\sim\mathbf{P}\} \subseteq$ a MC-∃C-PD set $\Gamma^*$

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If $\Gamma^*$ is MC-∃C-PD then $\Gamma^*$ is Q-C (11.4.8)

↓ ☑

$\Gamma \cup \{\sim\mathbf{P}\} \subseteq$ a Q-C set $\Gamma^*$

↓ ☑

$\Gamma \cup \{\sim\mathbf{P}\}$ is Q-C

↓ ☑

$\Gamma \nvDash \mathbf{P}$

An ES-variant of $\Gamma \cup \{\sim\mathbf{P}\}$ is C-PD

↓

Any ES-variant of $\Gamma \cup \{\sim\mathbf{P}\} \subseteq$ a MC-∃C-PD set $\Gamma^*$

- To prove: For any ES set Γe, there's an MC-∃C-PD set Γ* such that Γe ⊆ Γ*.
- So suppose Γe is ES and C-PD.
- We'll set out a procedure for building a big set around Γe, and then prove that the resulting set is MC-∃C-PD.
- As in our completeness proof of SD, the procedure will involve rules for constructing a series of sets Γ2, Γ3, etc.
- Γ1 will be Γ. And Γ* will be the union of all sets in the Γ-series.

An ES-variant of $\Gamma \cup \{\sim\mathbf{P}\}$ is C-PD

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Any ES-variant of $\Gamma \cup \{\sim\mathbf{P}\} \subseteq$ a MC-∃C-PD set $\Gamma^*$

- To build our sets, we use an enumeration of all PL sentences.
- The rules for building sets are three:
  - If $\Gamma i \cup \{\mathbf{P}i\}$ is IC-PD:
    - $\Gamma i{+}1$ is the same as $\Gamma i$.
  - If $\Gamma i \cup \{\mathbf{P}i\}$ is C-PD, and **Pi** is NOT of the form $(\exists\mathbf{x})\mathbf{Q}$:
    - $\Gamma i{+}1$ is $\Gamma i \cup \{\mathbf{P}i\}$
  - If $\Gamma i \cup \{\mathbf{P}i\}$ is C-PD, and **Pi** is of the form $(\exists\mathbf{x})\mathbf{Q}$:
    - $\Gamma i{+}1$ is $\Gamma i \cup \{\mathbf{P}i, \mathbf{Q(a/x)}\}$, where **a** is the alphabetically earliest constant not occurring in **Pi** or any member of $\Gamma i$

An ES-variant of $\Gamma \cup \{\sim\mathbf{P}\}$ is C-PD

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Any ES-variant of $\Gamma \cup \{\sim\mathbf{P}\} \subseteq$ a MC-∃C-PD set $\Gamma^*$

- E.g., suppose Γn = {Aa, Bb, (∀x)~Fx}
- And suppose in our enumeration, we have:

  **P**n = (∃x)Fx,

  **P**n+1 = Hb&Gb,

  **P**n+2 = (∃x)~Fx
- Then Γn+1 = Γn = {Aa, Bb, (∀x)~Fx}
- And Γn+2 = {Aa, Bb, (∀x)~Fx, Hb&Gb}
- And Γn+3 = {Aa, Bb, (∀x)~Fx, Hb&Gb, (∃x)~Fx, ~Fc}

An ES-variant of $\Gamma \cup \{\sim \mathbf{P}\}$ is C-PD

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Any ES-variant of $\Gamma \cup \{\sim \mathbf{P}\} \subseteq$ a MC-∃C-PD set $\Gamma^*$

- We can now prove that Γ*, the union of all sets in this sequence, is Maximally Consistent in PD and Existentially Complete.
- First, we'll show that Γ* is C-PD.
  - If it were IC-PD, a finite subset would be IC-PD, since every derivation is finite.
  - Every finite subset of Γ* is a subset of Γn for some n. (?)
  - And every Γn is C-PD. This we can prove by mathematical induction.

An ES-variant of $\Gamma \cup \{\sim\mathbf{P}\}$ is C-PD

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Any ES-variant of $\Gamma \cup \{\sim\mathbf{P}\} \subseteq$ a MC-∃C-PD set $\Gamma^*$

- Basis clause: Γ1 is C-PD by hypothesis.
- Inductive step: Suppose Γk is C-PD.
  - There are three possibilities for Γk+1.
    - (1) Γk+1 is Γk, in which case it's C-PD.
    - (2) Γk+1 is Γk ∪ {**Pi**}. Rule 2 requires it to be C-PD.
    - (3) Γk+1 is Γk ∪ {**Pi**, **Q(a/x)**}. Rule 3 requires Γk ∪ {**Pi**} to be C-PD, where **Pi** is of the form (∃**x**)**Q**.
      - But how do we know we can add **Q(a/x)** to Γk ∪ {**Pi**}?

An ES-variant of $\Gamma \cup \{\sim \mathbf{P}\}$ is C-PD

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Any ES-variant of $\Gamma \cup \{\sim \mathbf{P}\} \subseteq$ a MC-∃C-PD set $\Gamma^*$

- By assumption, **Pi** is of the form (∃**x**)**Q** and Γk ∪ {(∃**x**)**Q**} is C-PD.
- Assume Γk ∪ {(∃**x**)**Q**, **Q(a/x)**} is IC-PD.
- That means from Γk ∪ {(∃**x**)**Q**, **Q(a/x)**} we can derive **R** and **~R**.
- But that means given Γk ∪ {(∃**x**)**Q**} we can form a sub-derivation from assumption **Q(a/x)** to anything we want.
  - In particular, from **Q(a/x)** to ~(∃**x**)**Q**!
  - And then we can use ∃E to get ~(∃**x**)**Q** on the main scope line!
    - How do we know that's permitted?
  - We already have (∃**x**)**Q**, so we have derived a contradiction.

An ES-variant of $\Gamma \cup \{\sim\mathbf{P}\}$ is C-PD

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Any ES-variant of $\Gamma \cup \{\sim\mathbf{P}\} \subseteq$ a MC-∃C-PD set $\Gamma^*$

- So we've proven our inductive step. Now we have:

- Basis clause: $\Gamma 1$ is C-PD by hypothesis.
- Inductive step: If $\Gamma k$ is C-PD, then on any of the three ways $\Gamma k+1$ could be formed, $\Gamma k+1$ C-PD.
- Conclusion: For every n, $\Gamma n$ is C-PD.

An ES-variant of $\Gamma \cup \{\sim\mathbf{P}\}$ is C-PD

↓

Any ES-variant of $\Gamma \cup \{\sim\mathbf{P}\} \subseteq$ a MC-∃C-PD set $\Gamma^*$

- Now return to our proof that Γ* is C-PD.
- We said if Γ* were IC-PD, a finite subset would be IC-PD, since every derivation is finite.
- And every finite subset of Γ* is a subset of Γn for some n.
- Now, since we just showed that every Γn is C-PD, we know that any subset of any Γn is C-PD.
- So, since every finite subset of Γ* is a subset of some Γn, we know that any finite subset of Γ* is C-PD.
- So there's no derivation of any **R** and **~R** from Γ*, since that would have to be from a finite subset that was IC-PD.
- So Γ* is C-PD.

An ES-variant of $\Gamma \cup \{\sim\mathbf{P}\}$ is C-PD

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Any ES-variant of $\Gamma \cup \{\sim\mathbf{P}\} \subseteq$ a MC-∃C-PD set $\Gamma^*$

- Now we need to show that Γ* is maximal.
- Suppose the contrary: There's a **P**k∉Γ* s.t. Γ* ∪ {**P**k} is C-PD.
  - Since **P**k is a PL sentence, it occurs kth in our enumeration.
  - By the def. of our Γ-sequence, Γk+1 = Γk ∪ {**P**k} if that's C-PD.
  - Γk ∪ {**P**k} **is** C-PD.
    - Since if {**P**k} were inconsistent with Γk, it would be inconsistent with every superset of Γk, e.g. Γ*.
  - So Γk+1 = Γk ∪ {**P**k} (...perhaps with a substitution instance)
  - But that means **P**k∈Γk+1, so because Γk+1⊆Γ*, **P**k∈Γ*.
  - Contradiction.
- So there's no **P**k∉Γ* s.t. Γ* ∪ {**P**k} is C-PD. I.e. Γ* is maximal.

An ES-variant of $\Gamma \cup \{\sim\mathbf{P}\}$ is C-PD

↓

Any ES-variant of $\Gamma \cup \{\sim\mathbf{P}\} \subseteq$ a MC-∃C-PD set $\Gamma^*$

- Finally, let's show that Γ* is existentially complete.
  - I.e., that for each sentence in Γ* of the form **(∃x)Q**, there's a substitution instance of **(∃x)Q** in Γ*.
- So suppose **(∃x)Q** is in Γ*.
  - **(∃x)Q** is in our enumeration, at some position k.
  - Γk ∪ {**P**k}, i.e. Γk ∪ {**(∃x)Q**}, is either C-PD or IC-PD.
  - If it were IC-PD, then since **(∃x)Q** is in Γ*, Γ* would be IC-PD.
  - So Γk ∪ {**(∃x)Q**} is C-PD.

# Completeness

$\Gamma \nvdash$ **P**

↓ ☑

An ES-variant of $\Gamma \cup \{\sim$**P**$\}$ is C-PD

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Any ES-variant of $\Gamma \cup \{\sim$**P**$\} \subseteq$ a MC-∃C-PD set $\Gamma^*$

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If $\Gamma^*$ is MC-∃C-PD then $\Gamma^*$ is Q-C (11.4.8)

↓ ☑

$\Gamma \cup \{\sim$**P**$\} \subseteq$ a Q-C set $\Gamma^*$

↓ ☑

$\Gamma \cup \{\sim$**P**$\}$ is Q-C

↓ ☑

$\Gamma \nvDash$ **P**