## Honeycombs - In-plane behavior

- prismatic cells
- polymer, metal, ceramic honeycombs widely available
- used for sandwich structure cores, energy absorption, carriers for catalysts
- some natural materials (eg. wood, cork) can be idealized as honeycombs
- mechanisms of deformation + failure in hexagonal honeycombs parallel those in foams
    - simpler geometry (unit cell) - easier to analyze
- mechanisms of deformation in triangular honeycombs parallels those in 3D trusses (lattice materials)

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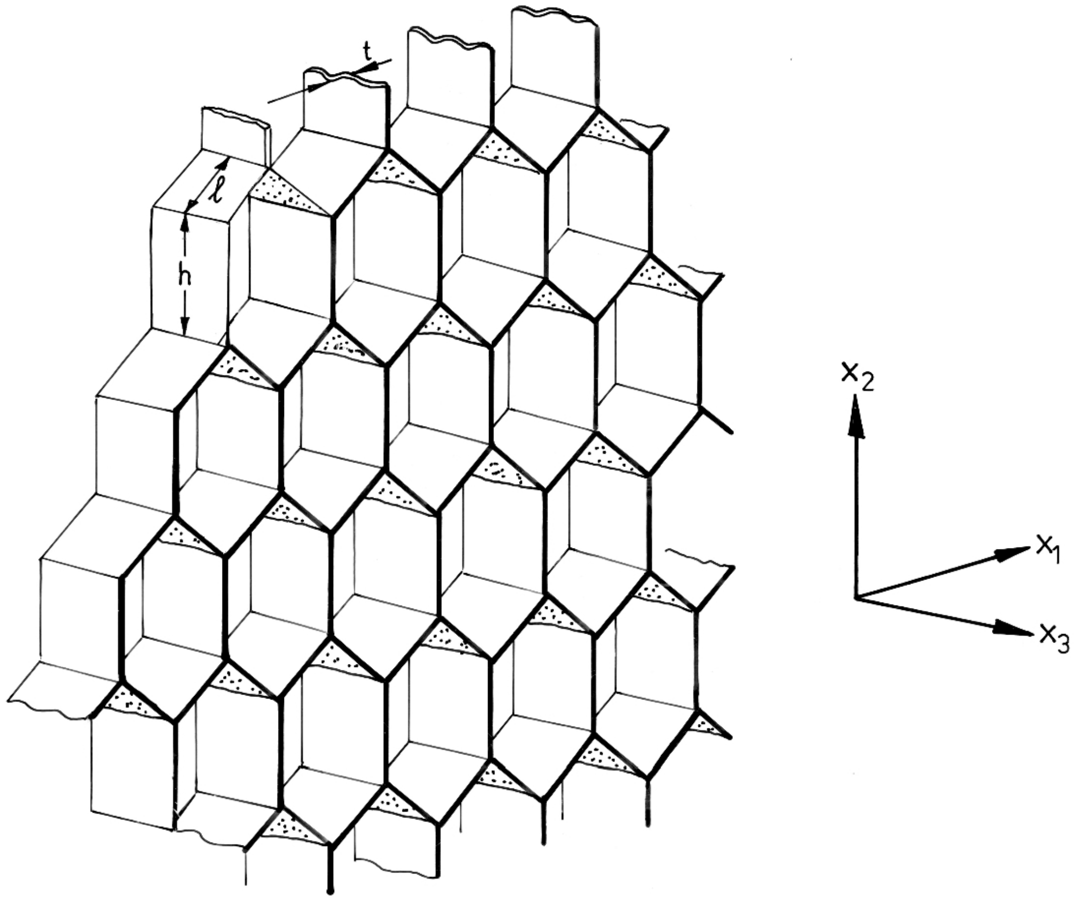
## Stress - Strain curves + Deformation behaviour : In-Plane

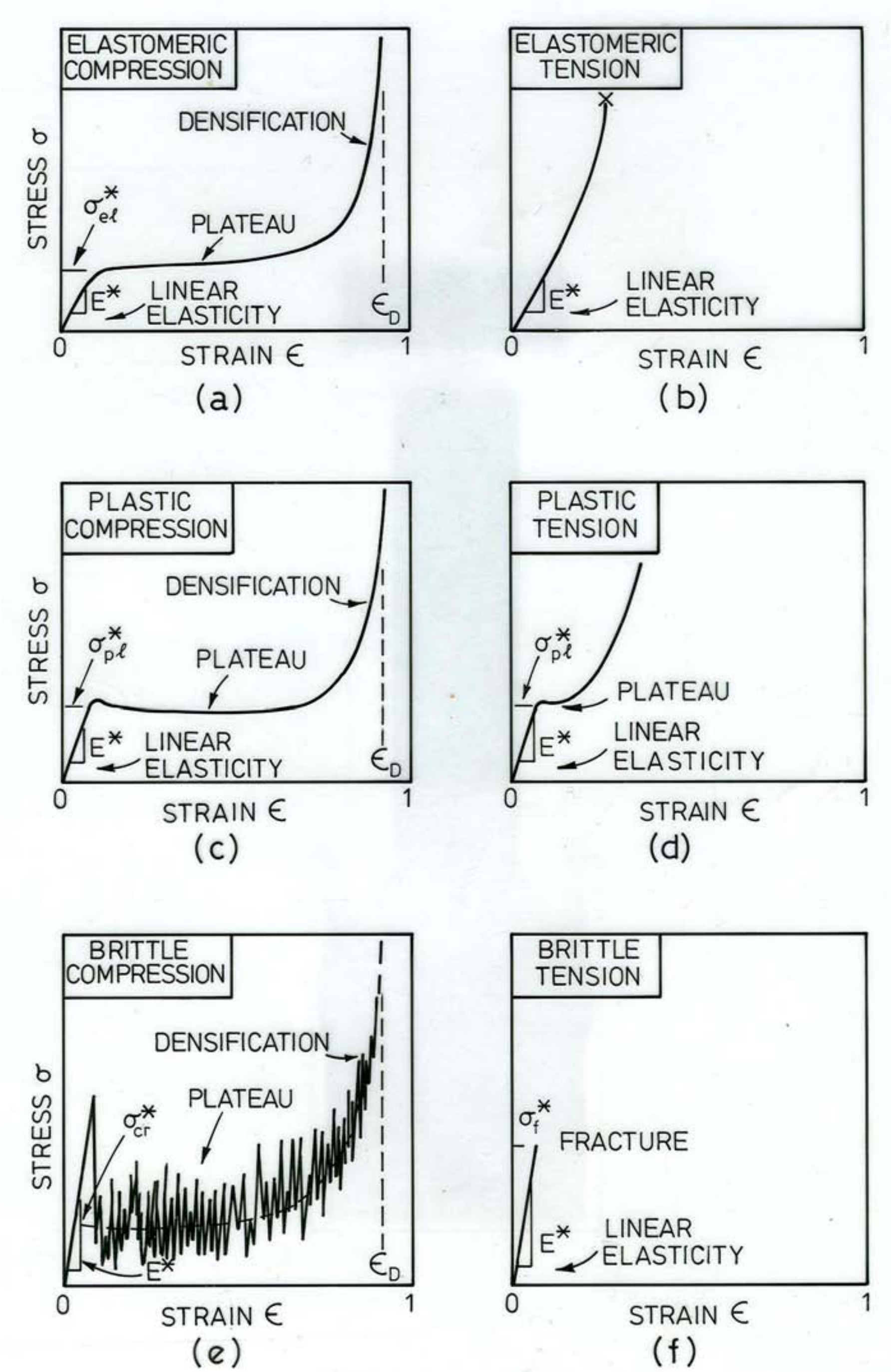
Compression

- 3 regimes
    - linear elastic - bending
    - stress plateau
        - buckling
        - yielding
        - brittle crushing
    - densification - cell walls touch
- increasing $t/l \Rightarrow E^* \uparrow \quad \sigma^* \uparrow \quad \epsilon_D \downarrow$

# Honeycomb Geometry




Gibson, L. J., and M. F. Ashby. *Cellular Solids: Structure and Properties*. 2nd ed. Cambridge University Press, 1997. Figure courtesy of Lorna Gibson and Cambridge University Press.

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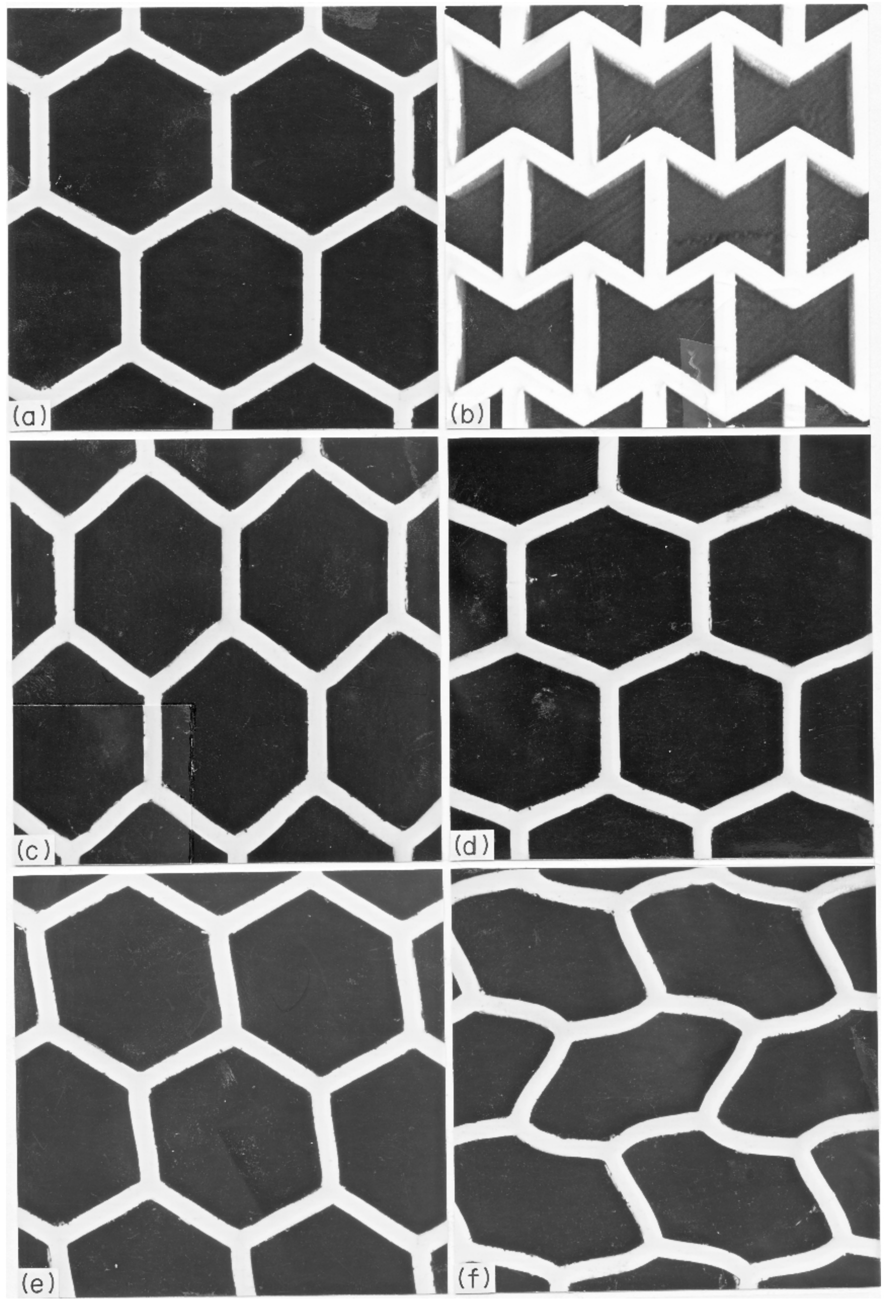
# Deformation mechanisms



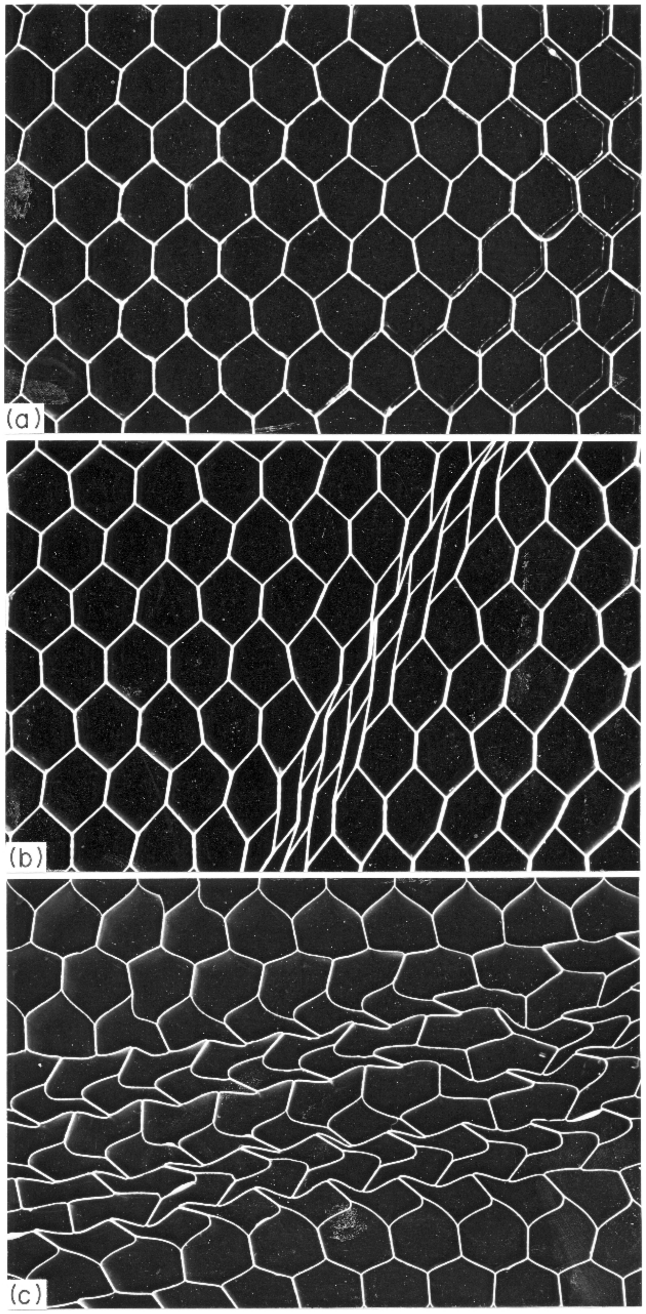

Bending
$X_1$ Loading

Bending
$X_2$ Loading

Bending
Shear

Buckling

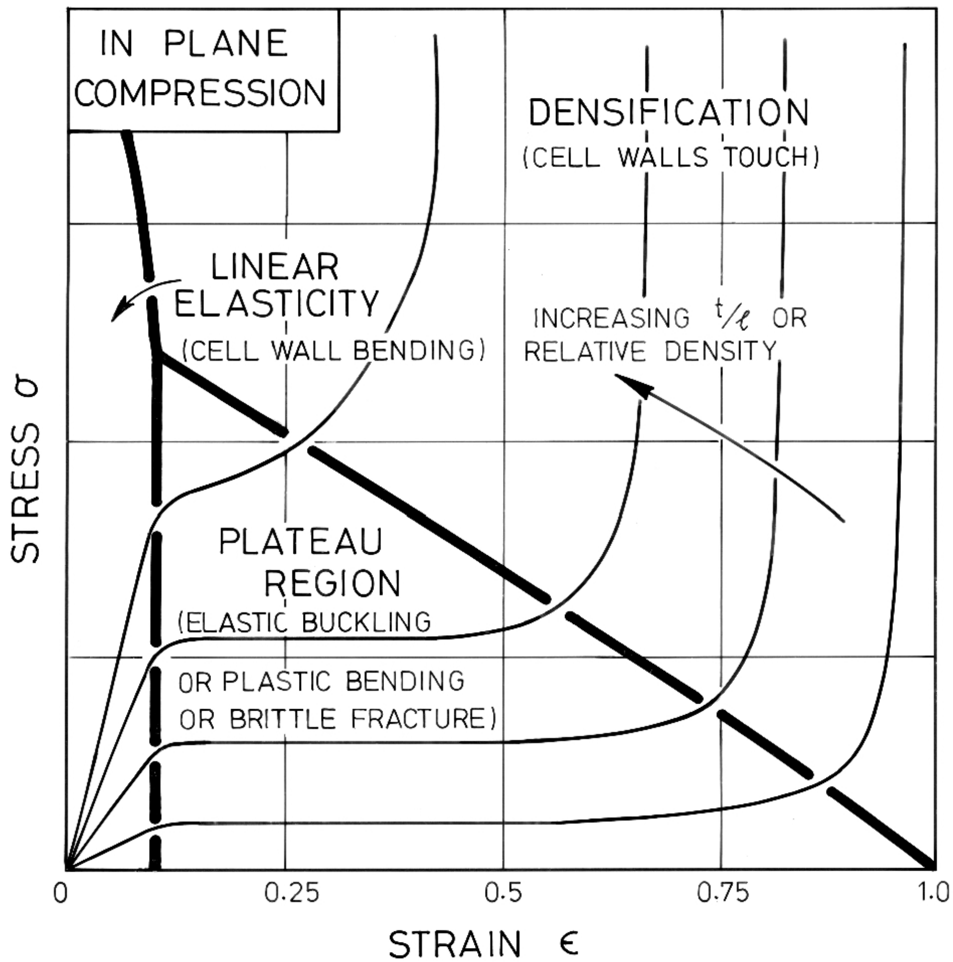
Gibson, L. J., and M. F. Ashby. *Cellular Solids: Structure and Properties*. 2nd ed. Cambridge University Press, 1997. Figure courtesy of Lorna Gibson and Cambridge University Press.




Plastic collapse in an aluminum honeycomb

Gibson, L. J., and M. F. Ashby. *Cellular Solids: Structure and Properties*. 2nd ed. Cambridge University Press, 1997. Figure courtesy of Lorna Gibson and Cambridge University Press.

# Stress-Strain Curve




Gibson, L. J., and M. F. Ashby. *Cellular Solids: Structure and Properties*. 2nd ed. Cambridge University Press, 1997. Figure courtesy of Lorna Gibson and Cambridge University Press.

## tension

- linear elastic - bending
- stress plateau - exists only if cell walls yield
  - no buckling in tension
  - brittle honeycombs fracture in tension

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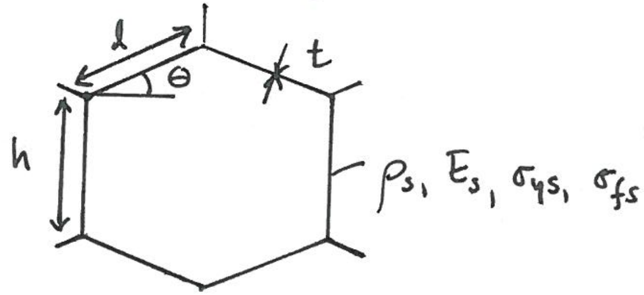
## Variables affecting honeycomb properties

- relative density $\frac{\rho^*}{\rho_s} = \frac{t/l\,(h/l + 2)}{2\cos\theta\,(h/l + \sin\theta)} = \frac{2}{\sqrt{3}}\frac{t}{l}$ regular hexagons
- solid cell wall properties: $\rho_s$, $E_s$, $\sigma_{ys}$, $\sigma_{fs}$
- cell geometry: $h/l$, $\theta$



## In-plane properties

assumptions:

- $t/l$ small ($\rho^*/\rho_s$ small) - neglect axial + shear contribution to def'm
- deformations small - neglect changes in geometry
- cell wall - linear elastic, isotropic

symmetry

- honeycombs are orthotropic - rotate 180° about each of three mutually perpendicular axes & structure is the same

## Linear elastic deformation

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & 0 & 0 & 0 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & -1/E_3 & 0 & 0 & 0 \\ & & & 1/G_{23} & 0 & 0 \\ & \text{(symmetric)} & & & 1/G_{13} & 0 \\ & & & & & 1/G_{12} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}$$

- Matrix notation:

$$\epsilon_1 = \epsilon_{11} \quad \epsilon_4 = \gamma_{23} \quad \sigma_1 = \sigma_{11} \quad \sigma_4 = \sigma_{23}$$
$$\epsilon_2 = \epsilon_{22} \quad \epsilon_5 = \gamma_{13} \quad \sigma_2 = \sigma_{22} \quad \sigma_5 = \sigma_{13}$$
$$\epsilon_3 = \epsilon_{33} \quad \epsilon_6 = \gamma_{12} \quad \sigma_3 = \sigma_{33} \quad \sigma_6 = \sigma_{12}$$

- in-plane $(x_1 - x_2)$ : 4 independent elastic constants:

$$E_1 \quad E_2 \quad \nu_{12} \quad G_{12}$$

and compliance matrix symmetric $-\frac{\nu_{12}}{E_1} = -\frac{\nu_{21}}{E_2}$ (reciprocal relation)

[ notation for Poisson's ratio: $\nu_{ij} = -\frac{\epsilon_j}{\epsilon_i}$ ]

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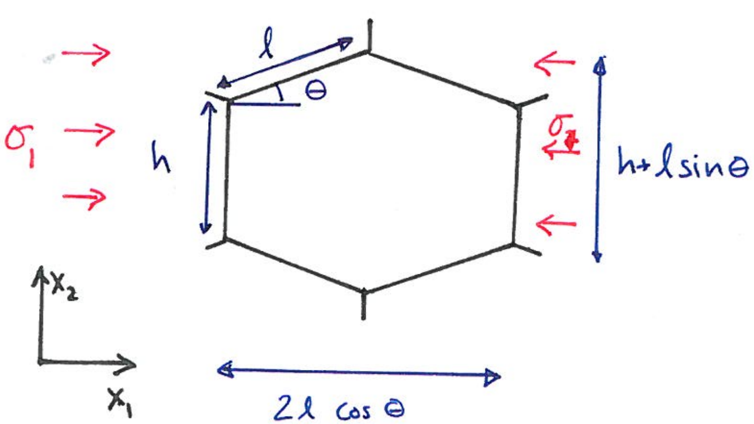
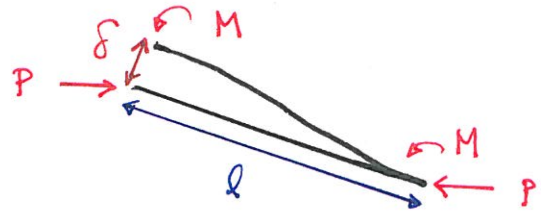
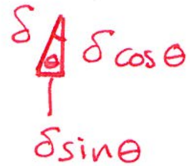
Young's modulus in $x_1$ direction



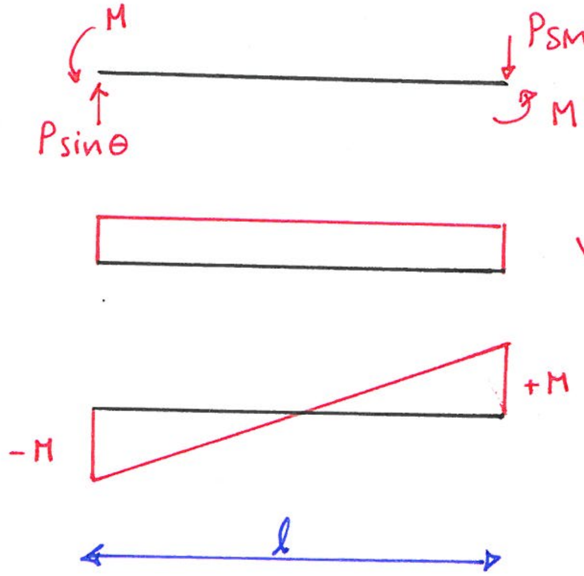




$$\sigma_1 = \frac{P}{(h + l\sin\theta)b}$$

$$\epsilon_1 = \frac{\delta\sin\theta}{l\cos\theta}$$

unit cell in $x_1$ direction: $2l\cos\theta$

# In-Plane Deformation: Linear Elasticity




V diagram

M diagram: 2 cantilevers of length $l/2$

$$\delta = 2 \cdot \frac{P\sin\theta\,(l/2)^3}{3 E_s I}$$

$$= \frac{2\,P l^3 \sin\theta}{24\, E_s I}$$

$$\delta = \frac{P l^3 \sin\theta}{12\, E_s I} \qquad I = \frac{bt^3}{12}$$

Combining: $$E_1^* = \frac{\sigma_1}{\epsilon_1} = \frac{P}{(h + l\sin\theta)\, b} \quad \frac{l\cos\theta}{\delta \sin\theta}$$

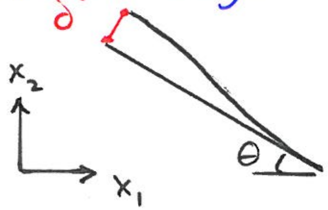
$$= \frac{\cancel{P}}{(h + l\sin\theta)\cancel{b}} \quad \frac{l\cos\theta}{\cancel{P}\, l^3 \sin^2\theta}\; \cancel{12}\, E_s \frac{\cancel{b}\, t^3}{\cancel{12}}$$

$$\boxed{E_1^* = \underbrace{E_s}_{\text{solid property}} \underbrace{\left(\frac{t}{l}\right)^3}_{\text{relative density}} \underbrace{\frac{\cos\theta}{(h/l + \sin\theta)\sin^2\theta}}_{\text{cell geometry}}} = \frac{4}{\sqrt{3}} \left(\frac{t}{l}\right)^3 E_s \quad \text{regular hexagons}$$

$h/l = 1 \quad \theta = 30°$

## Poisson's ratio for loading in $x_1$ direction

$$\nu_{12}^* = -\frac{\epsilon_2}{\epsilon_1}$$





$$\epsilon_1 = -\frac{\delta \sin\theta}{l\cos\theta} \quad \text{(shortens)} \qquad \epsilon_2 = \frac{\delta\cos\theta}{h + l\sin\theta} \quad \text{(lengthens)}$$

$$\nu_{12}^* = -\frac{\delta\cos\theta}{h + l\sin\theta}\left(\frac{l\cos\theta}{-\delta\sin\theta}\right) = \frac{\cos^2\theta}{(h/l + \sin\theta)\sin\theta}$$

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- $\nu_{12}^*$ depends ONLY on cell geometry ($h/l$, $\theta$), not on $E_s$, $t/l$
- regular hexagonal cells: $\nu_{12}^* = 1$
- $\nu$ can be negative for $\theta < 0$

  eg. $h/l = 2 \quad \theta = -30° \quad \nu_{12}^* = \dfrac{3/4}{(3/2)(-1/2)} = -1$

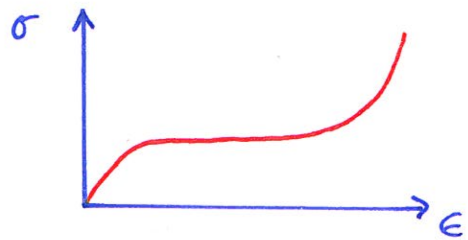
## $E_2^*$ $\nu_{21}^*$ $G_{12}^*$

- can be found in similar way; results in book.

## Compressive strength (plateau stress)

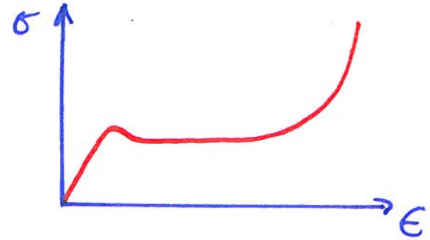
- cell collapse by:

(1) elastic buckling



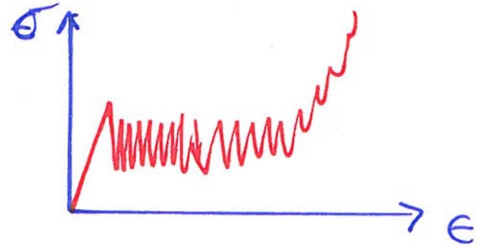

- buckling of vertical struts throughout honeycomb

(2) plastic yielding




- localization of yield
- as def'm progresses, propagation of failure band

(3) brittle crushing




- peaks + valleys correspond to fracture of individual cell walls

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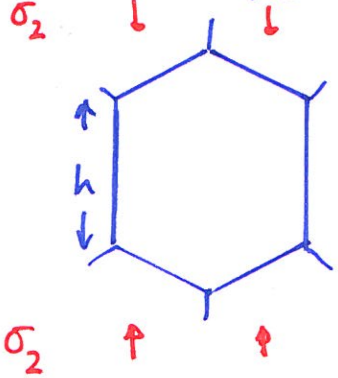
## Plateau stress: elastic buckling, $\sigma_{el}^*$

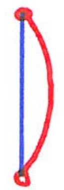
- elastomeric honeycombs - cell collapse by elastic buckling of walls of length $h$ when loaded in $x_2$ direction
- no buckling for $\sigma_1$; bending of inclined walls goes to densification.



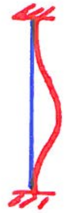

Euler buckling load

$$P_{cr} = \frac{n^2 \pi^2 E_s I}{h^2}$$

$n$ = end constraint factor



pin-pin
$n = 1$



fixed-fixed
$n = 2$

# Elastic Buckling

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- here, constraint $n$ depends on stiffness of adjacent inclined members
- can find by elastic line analysis (see appendix if interested)
- rotational stiffness at ends of column, $h$, matched to rotational stiffness of inclined members
- find $h/l = 1 \quad 1.5 \quad 2$

$n = 0.686 \quad 0.760 \quad 0.860$

and $$(\sigma^*_{el})_2 = \frac{P_{cr}}{2l\cos\theta\, b} = \frac{n^2\pi^2 E_s}{h^2\, 2l\cos\theta\, b}\frac{bt^3}{12}$$

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$$\boxed{(\sigma^*_{el})_2 = \frac{n^2\pi^2}{24} E_s \frac{(t/l)^3}{(h/l)^2\cos\theta}}$$

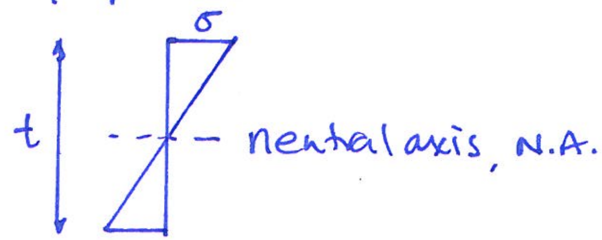
regular hexagons: $(\sigma^*_{el})_2 = 0.22\, E_s (t/l)^3$

& since $E^*_2 = 4/\sqrt{3}\, E_s (t/l)^3 = 2.31\, E_s (t/l)^3$

strain at buckling $(\epsilon^*_{el})_2 = 0.10$, for regular hexagons independent of $E_s$, $t/l$

## Plateau stress: plastic yielding, $\sigma^*_{pl}$

- failure by yielding in cell walls
- yield strength of cell walls $= \sigma_{ys}$
- plastic hinge forms when cross-section fully yielded
- beam theory - linear elastic $\sigma = \frac{My}{I}$

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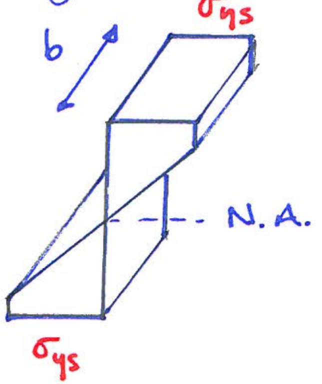
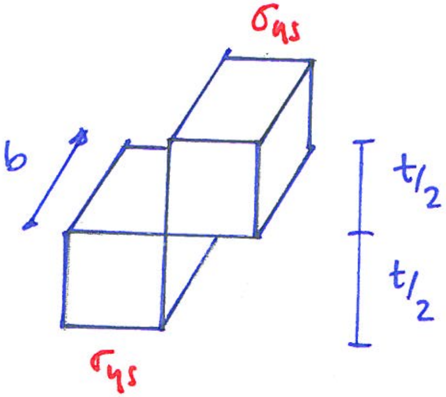
- once stress at outer fiber $= \sigma_{ys}$, yielding begins & then progresses through the section, as the load increases

as P ↑




- when section fully yielded (right fig.), form plastic "hinge"
- section rotates, like a pin

# Plastic Collapse

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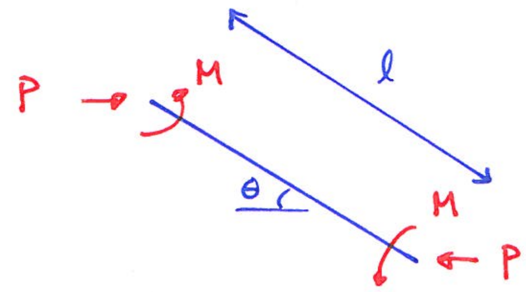
- moment at formation of plastic hinge (plastic moment, $M_p$):

$$M_p = \left(\sigma_{ys}\, b \frac{t}{2}\right)\left(\frac{t}{2}\right) = \frac{\sigma_{ys}\, b t^2}{4}$$

- applied moment, from applied stress




$$2 M_{app} - P l \sin\theta = 0$$

$$M_{app} = \frac{P l \sin\theta}{2}$$

$$\sigma_1 = \frac{P}{(h + l\sin\theta)\, b} \qquad M_{app} = \sigma_1 (h + l\sin\theta)\, b\, \frac{l\sin\theta}{2}$$

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- plastic collapse of honeycomb at $(\sigma^*_{pl})_1$ when $M_{app} = M_p$

$$(\sigma^*_{pl})_1 (h + l\sin\theta)\, \cancel{b}\, \frac{l\sin\theta}{\cancel{2}} = \sigma_{ys} \frac{\cancel{b} t^2}{\cancel{4}\, 2}$$

$$\boxed{(\sigma^*_{pl})_1 = \sigma_{ys}\left(\frac{t}{l}\right)^2 \frac{1}{2(h/l + \sin\theta)\sin\theta}}$$

regular hexagons: $(\sigma^*_{pl})_1 = \frac{2}{3}\sigma_{ys}\left(\frac{t}{l}\right)^2$

similarly, $(\sigma^*_{pl})_2 = \sigma_{ys}\left(\frac{t}{l}\right)^2 \frac{1}{2\cos^2\theta}$

- can do similar

- for thin-walled honeycombs, elastic buckling can precede plastic collapse (for $\sigma_2$)
- elastic buckling stress = plastic collapse stress $(\sigma^*_{el})_2 = (\sigma^*_{pl})_2$

$$\frac{n^2 \pi^2}{24} E_s \frac{(t/l)^3}{(h/l)^2 \cos\theta} = \frac{\sigma_{ys} (t/l)^2}{2 \cos^2\theta}$$

$$(t/l)_{critical} = \frac{12 \, (h/l)^2}{n^2 \pi^2 \cos\theta} \left( \frac{\sigma_{ys}}{E_s} \right)$$

regular hexagons: $(t/l)_{critical} = 3 \dfrac{\sigma_{ys}}{E_s}$.

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- e.g. metals $\sigma_{ys}/E_s \sim .002$ $(t/l)_{crit} \sim 0.6\%$
  - most metal honeycomb denser than this

  polymers $\sigma_{ys}/E_s \sim 3\text{-}5\%$ $(t/l)_{crit} \sim 10\text{-}15\%$
  - low density polymers with yield point may buckle before yield.

Plateau stress: brittle crushing, $(\sigma_{cr}^*)_1$

- ceramic honeycombs - fail in brittle manner
- cell wall bending - stress reaches modulus of rupture - wall fracture

loading in $x_1$ direction:

$$P = \sigma_1 (h + l\sin\theta) b \qquad \sigma_{fs} = \text{Modulus of rupture of cell wall}$$

$$M_{\substack{max \\ applied}} = \frac{Pl\sin\theta}{2} = \frac{\sigma_1 (h + l\sin\theta) bl\sin\theta}{2}$$

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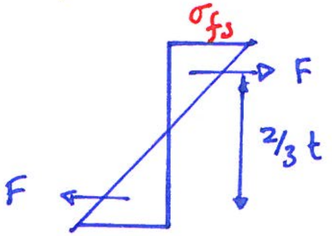
Moment at fracture, $M_f$

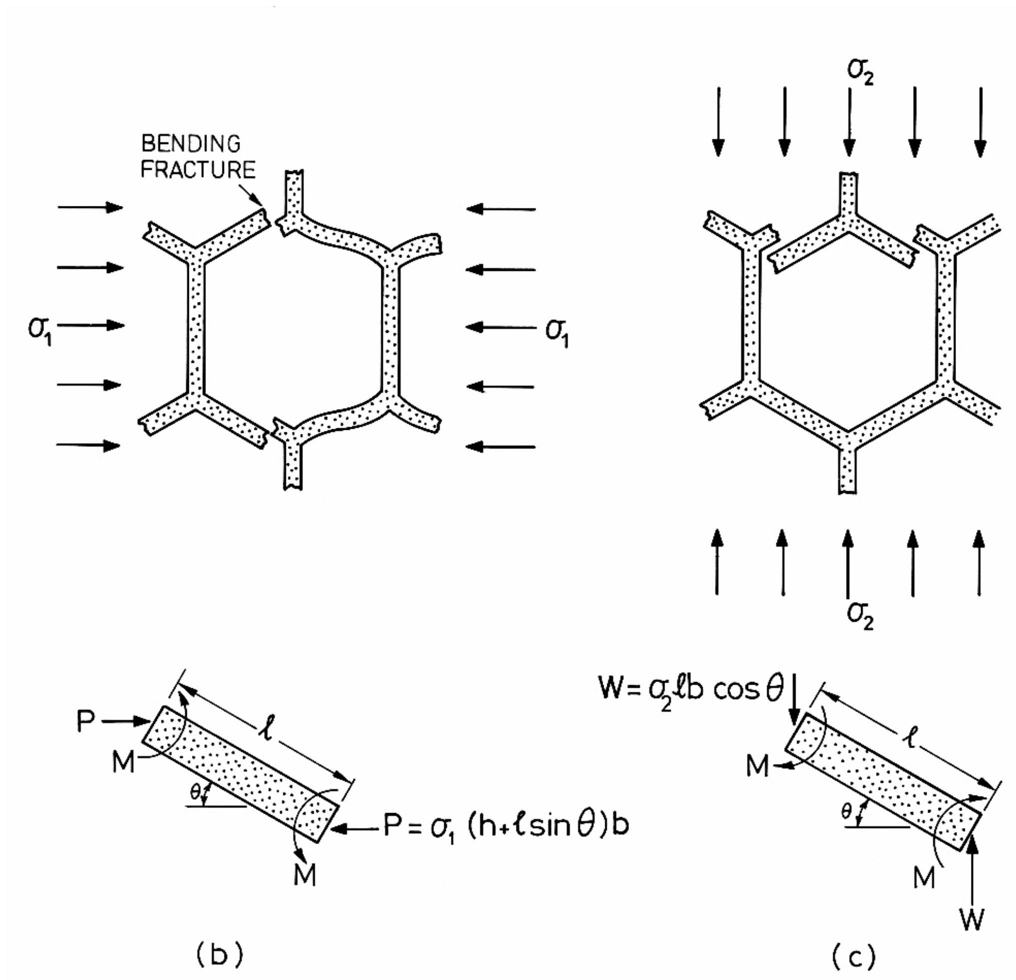



$$M_f = \left(\frac{1}{2}\sigma_{fs} b \frac{t}{2}\right)\left(\frac{2}{3}t\right) = \frac{\sigma_{fs} b t^2}{6}$$

$$\boxed{(\sigma_{cr}^*)_1 = \sigma_{fs}\left(\frac{t}{l}\right)^2 \frac{1}{3(h/l + \sin\theta)\sin\theta}}$$

regular hexagons: $(\sigma_{cr}^*)_1 = \frac{4}{9}\sigma_{fs}\left(\frac{t}{l}\right)^2$

# Brittle Crushing

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## Tension

- no elastic buckling
- plastic plateau stress approx same in tension + compression
  (small geometric difference due to deformation)
- brittle honeycombs: fast fracture

## Fracture toughness

assume:
- crack length large relative to cell size (continuum assumption)
- axial forces can be neglected
- cell wall material has constant modulus of rupture, $\sigma_{fs}$

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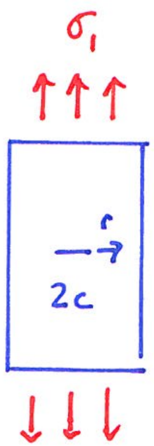
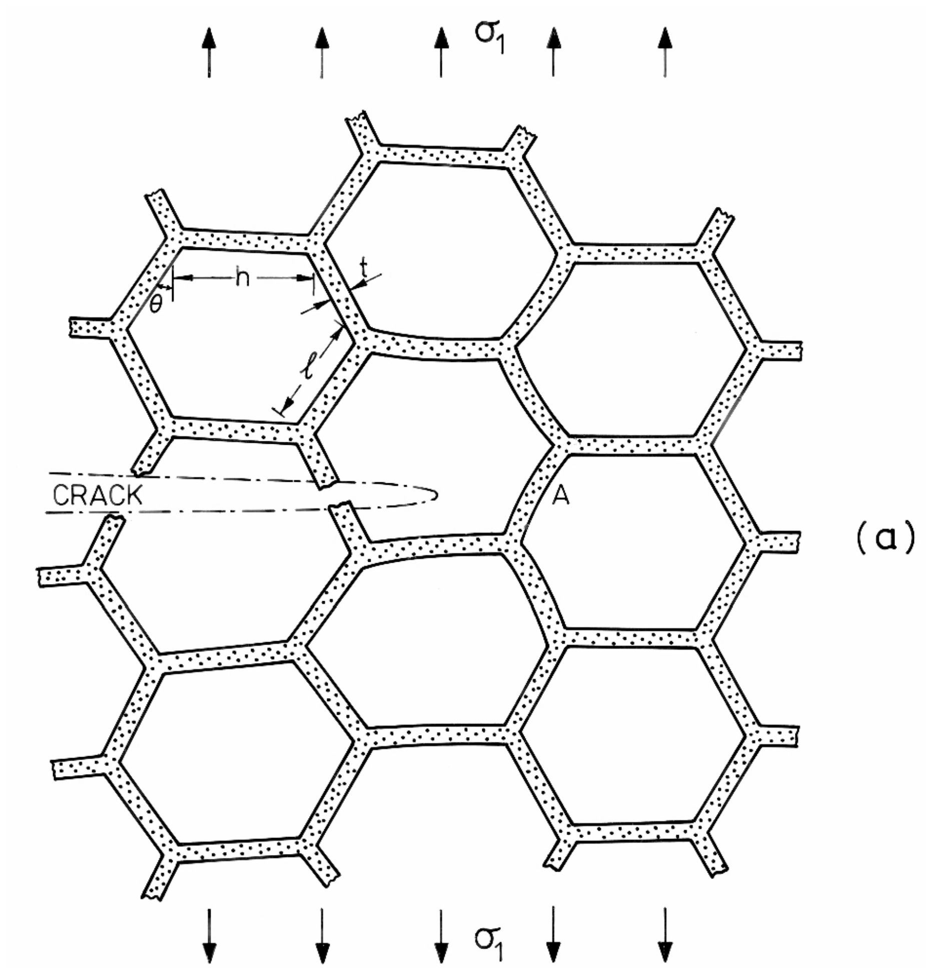
continuum: crack of length $2c$ in a linear elastic solid material normal to a remote tensile stress $\sigma_1$ creates a local stress field at the crack tip




$$\sigma_{local} = \sigma_l = \frac{\sigma_1 \sqrt{\pi c}}{\sqrt{2\pi r}}$$

# Fracture Toughness

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honeycomb: cell walls bent - fail when applied moment = fracture moment

$M_{app} \propto P\ell$ on wall A $\qquad M_f \propto \sigma_{fs}\, b t^2$

$$M_{app} \propto P\ell \propto \sigma_\ell\, \ell^2 b \propto \frac{\sigma_1 \sqrt{c}\, \ell^2 b}{\sqrt{\ell}} \propto \sigma_{fs}\, b t^2$$

$$(\sigma_f^*)_1 \propto \sigma_{fs} \left(\frac{t}{\ell}\right)^2 \sqrt{\frac{\ell}{c}}$$

$$\boxed{K_{Ic}^* = \sigma_f^* \sqrt{\pi c} = C\, \sigma_{fs} \left(t/\ell\right)^2 \sqrt{\ell}}$$

depends on cell size, $\ell$!

C = constant

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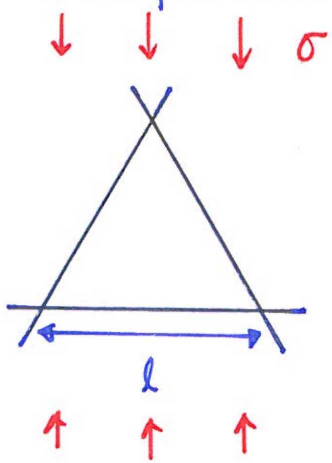
Summary: hexagonal honeycombs, in-plane properties

- linear elastic moduli: $E_1^*$ $E_2^*$ $\nu_{12}^*$ $G_{12}^*$
- plateau stresses (compression) $(\sigma_{el}^*)_2$ elastic buckling

  $\sigma_{pl}^*$ plastic yield

  $\sigma_{cr}^*$ brittle crushing
- fracture toughness (tension) $K_{Ic}^*$ brittle fracture

# Honeycombs: In-plane behaviour - triangular cells

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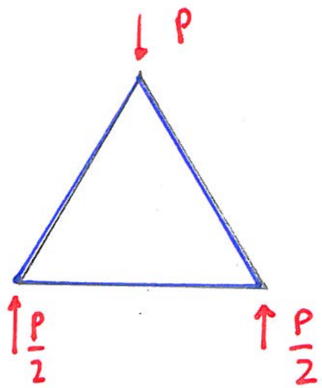
- triangulated structures - trusses
- can analyze as pin-jointed (no moment @ joints)
- forces in members all axial (no bending)
- if joints fixed + include bending, difference ~ 2%
- force in each member proportional to P



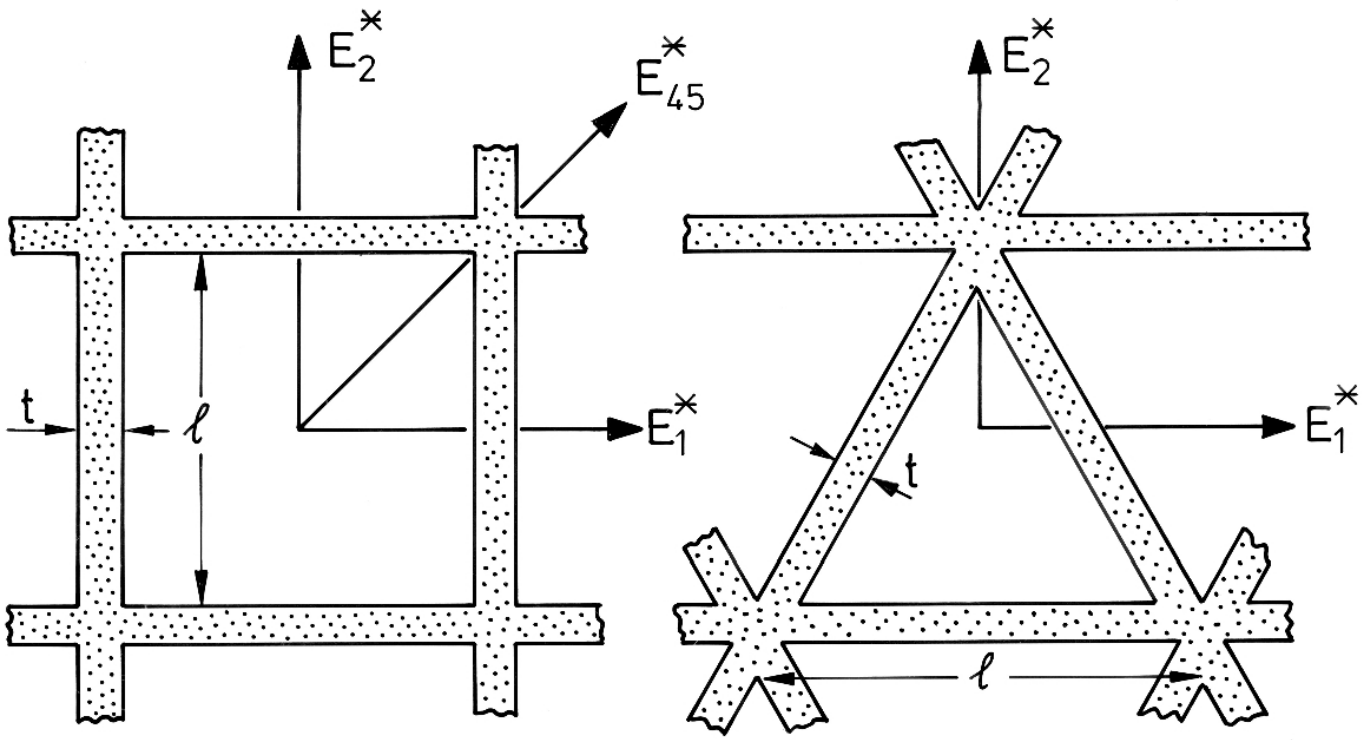

$\sigma \propto \frac{P}{l b}$ $\quad \epsilon \propto \frac{\delta}{l}$ $\quad \delta \propto \frac{P l}{A E_s}$ (axial shortening: Hooke's law)

$$E^* \propto \frac{\sigma}{\epsilon} \propto \frac{P}{lb} \frac{l}{\delta} \propto \frac{P}{b} \frac{b t E_s}{P l} \propto E_s \left(\frac{t}{l}\right)$$

$$E^* = C E_s (t/l)$$

exact calculation: $E^* = 1.15\, E_s (t/l)$ for equilateral triangle

# Square and Triangular Honeycombs

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3.054 / 3.36 Cellular Solids: Structure, Properties and Applications
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