Open-cell foams

- stress-strain curves : deformation + failure mechanisms
- compression - 3 regimes - linear elastic - bending
  - stress plateau - cell collapse by buckling
    yielding
    crushing
  - densification
- tension - no buckling
  - yielding can occur
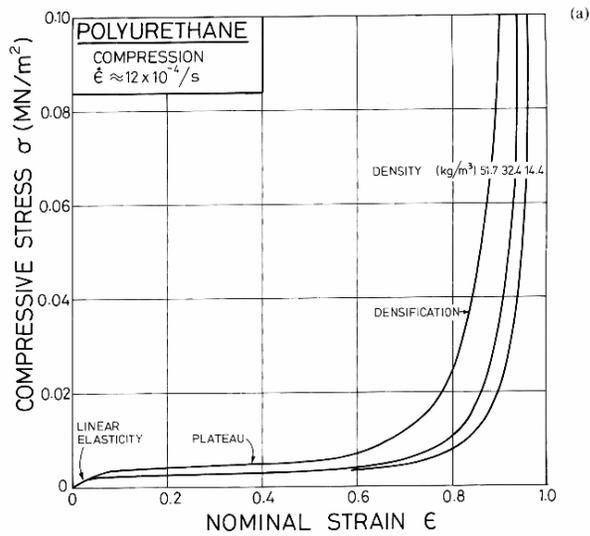 - brittle fracture

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Linear elastic behaviour

- initial linear elasticity - bending of cell edges (small t/l)

- as t/l ↑, axial deformation becomes more significant

- consider dimensional argument, which models mechanism of deformation + failure, but not cell geometry

- consider cubic cell, square cross-section members of area t^2, length, l
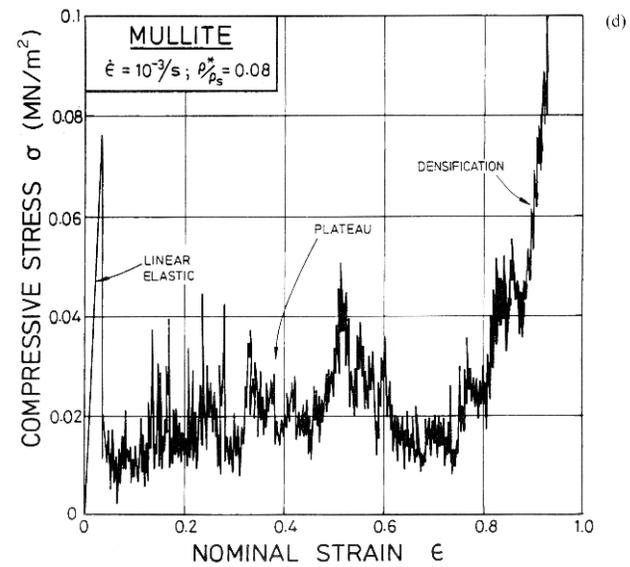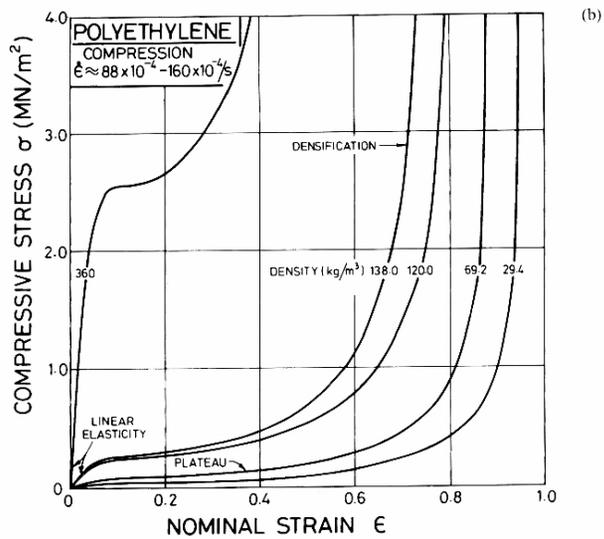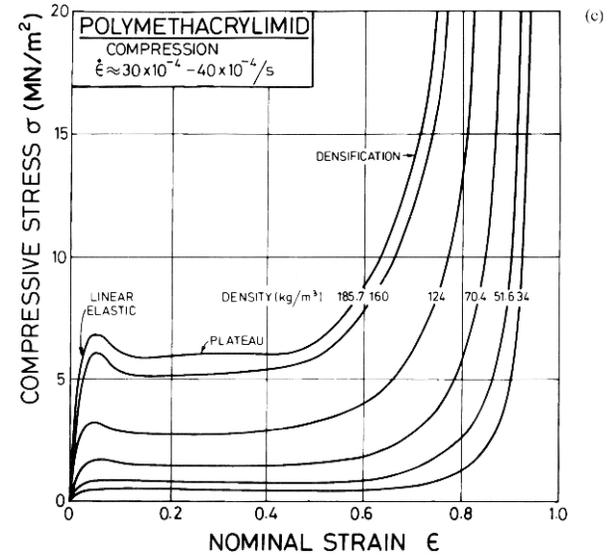
Gibson, L. J., and M. F. Ashby. *Cellular Solids: Structure and Properties*. 2nd ed. Cambridge University Press, © 1997. Figures courtesy of Lorna Gibson and Cambridge University Press.

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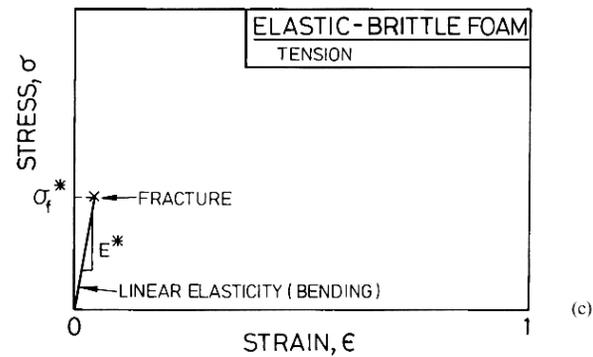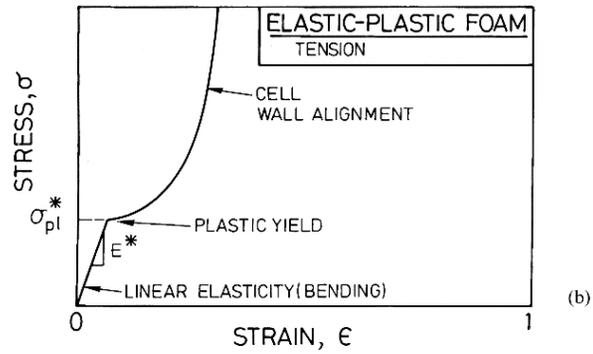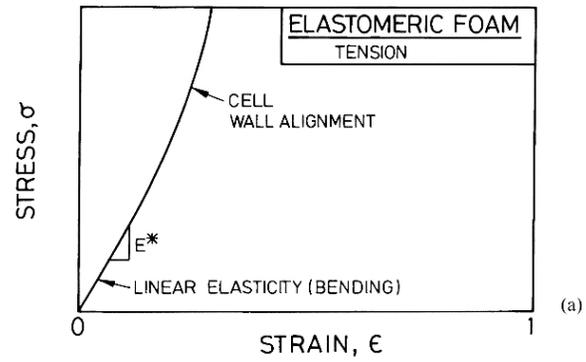
Foams: Bending, Buckling

Foams: Plastic Hinges
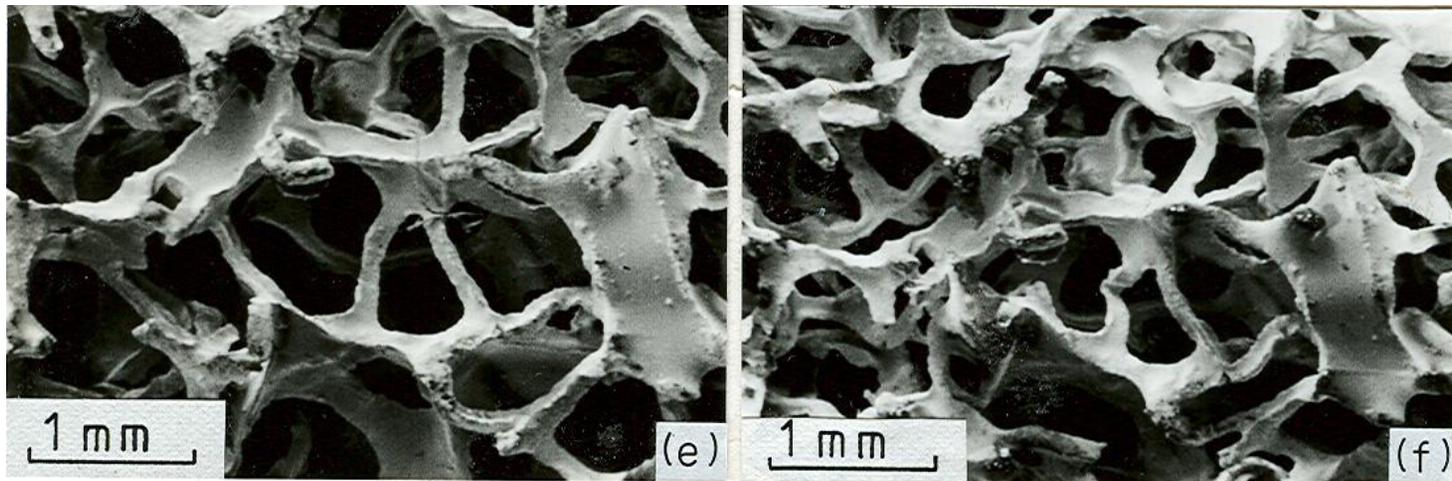

Gibson, L. J., and M. F. Ashby. *Cellular Solids: Structure and Properties*. 2nd ed. Cambridge University Press, © 1997. Figure courtesy of Lorna Gibson and Cambridge University Press.

Foams: Cell Wall Fracture

Figure removed due to copyright restrictions. See Fig. 3: Gibson, L. J., and M. F. Ashby. "The Mechancis of Three-Dimensional Cellular Materials." *Proceedings of The Royal Society of London A* 382 (1982): 43-59.

- regardless of specific cell geometry chosen:

$$\rho^*/\rho_s \propto (t/\ell)^2 \qquad I \propto t^4$$

$$\sigma \propto F/\ell^2 \qquad \epsilon \propto \delta/\ell \qquad \delta \propto F\ell^3/E_s I$$

$$E^* \propto \frac{\sigma}{\epsilon} \propto \frac{F}{\ell^2}\frac{\ell}{\delta} \propto \frac{F}{\ell}\frac{E_s t^4}{F\ell^3} \propto E_s\left(\frac{t}{\ell}\right)^4 \propto E_s\left(\frac{\rho^*}{\rho_s}\right)^2$$

$$\boxed{E^* = C_1 E_s \left(\rho^*/\rho_s\right)^2}$$

C_1 includes all geometrical constants.

Data: $C_1 \approx 1$

- data suggest $C_1 = 1$

- analysis of open cell tetrakaidecahedral cells with Plateau borders
  gives $C_1 = 0.98$

- shear modulus $\boxed{G^* = C_2 E_s \left(\rho^*/\rho_s\right)^2}$ $\qquad C_2 \sim 3/8$ if foam isotropic
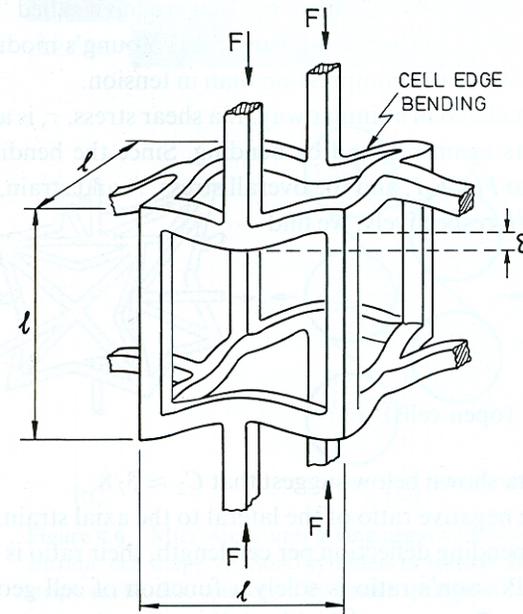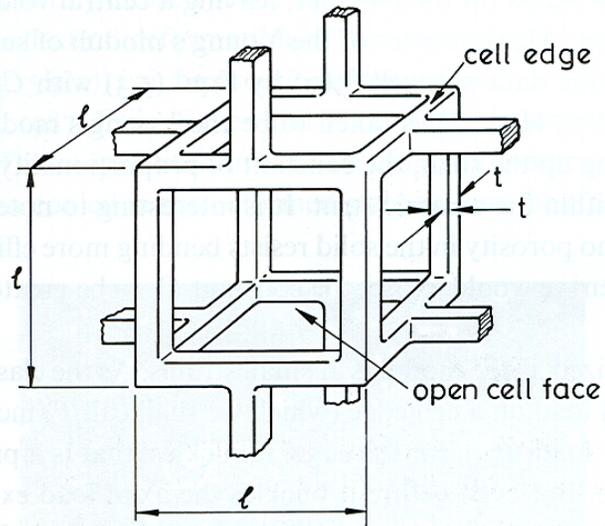
  isotropy: $G = \dfrac{E}{2(1+\nu)}$

- Poisson's ratio $\nu^* = \dfrac{E}{2G} - 1 = \dfrac{C_1}{2C_2} - 1 =$ constant, independent of
  $E_s, t/\ell$

$$\boxed{\nu^* = C_3}$$

(analogous to honeycombs in-plane)

Foam: Edge Bending

Poisson's ratio

- can make negative Poisson's ratio foams

- invert cell angles (analogous to honeycomb)

- eg. thermoplastic foams - load hydrostatically.

  + heat to $T > T_g$, then cool + release load

  so that edges of cell permanently point inward.

Closed-cell foams

- edge bending as for open-cell foams

- also: face stretching + gas compression

- polymer foams: surface tension draws material to edges during processing

  - define t_e, t_f in figure

- apply F to the cubic structure

Negative Poisson's Ratio
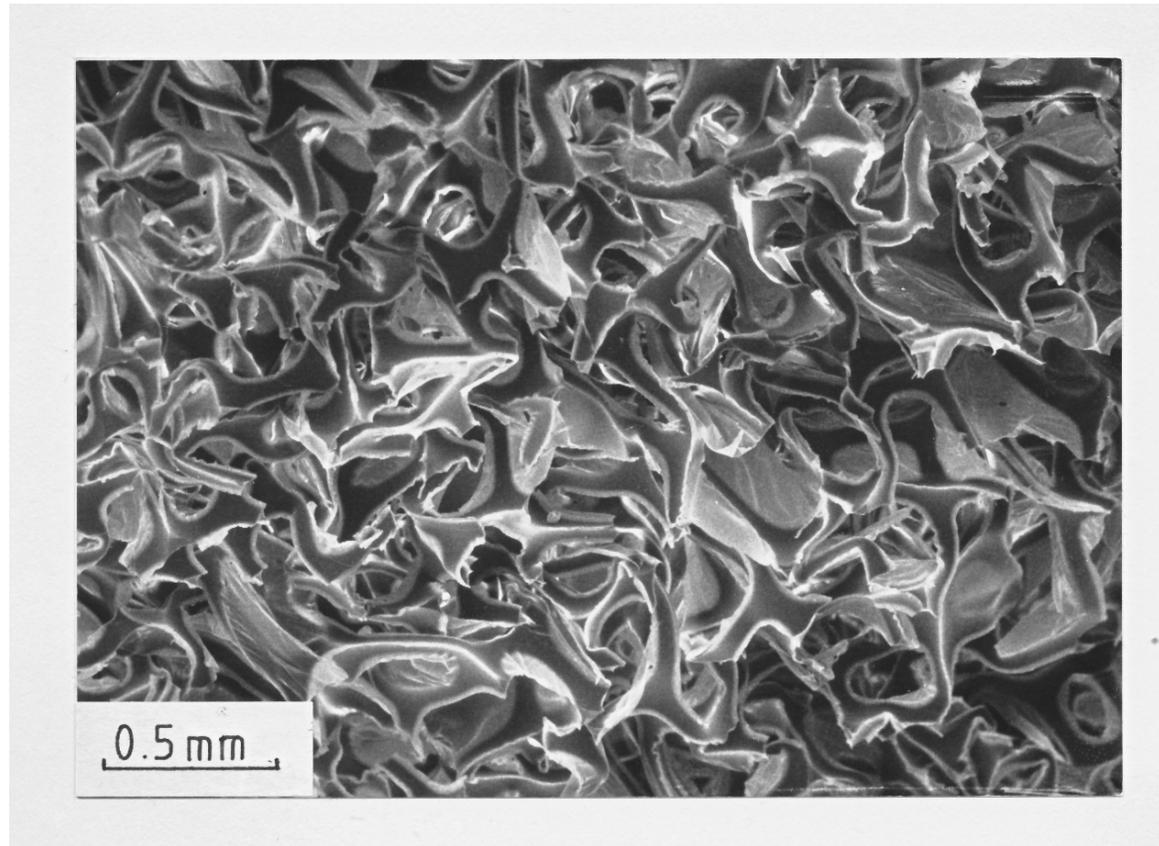
· external work done α $F\delta$

· internal work bending edges α $\dfrac{F_e}{\delta_e} \delta_e^2$ α $\dfrac{E_s I \delta^2}{\ell^3}$

· internal work stretching faces α $\sigma_f \epsilon_f V_f$ α $E_s \epsilon_f^2 V_f$ α $E_s \left(\dfrac{\delta}{\ell}\right)^2 t_f \ell^2$

$$\therefore F\delta = \alpha \frac{E_s t_e^4 \delta^2}{\ell^3} + \beta E_s \left(\frac{\delta}{\ell}\right)^2 t_f \ell^2$$

$$E^* \alpha \frac{F}{\ell^2} \frac{\ell}{\delta} \qquad \Rightarrow F \alpha E^* \delta \ell$$

$$\therefore E^* \delta^2 \ell = \alpha \frac{E_s t_e^4}{\ell^3} \delta^2 + \beta E_s \left(\frac{\delta}{\ell}\right)^2 t_f \ell^2$$

$$E^* = \alpha E_s \left(t_e/\ell\right)^4 + \beta E_s \left(t_f/\ell\right)$$

Note: open cells, uniform t :

$$\rho^*/\rho_s \alpha \left(t/\ell\right)^2$$

closed cells, uniform t

$$\rho^*/\rho_s \alpha \left(t/\ell\right)$$

if ϕ = volume fraction of solid in cell edges:

$$t_e/\ell = C \phi^{1/2} \left(\rho^*/\rho_s\right)^{1/2}$$

$$t_f/\ell = C' (1-\phi) \left(\rho^*/\rho_s\right)$$

$$\boxed{\frac{E^*}{E_s} = C_1 \phi^2 \left(\frac{\rho^*}{\rho_s}\right)^2 + C_1' (1-\phi) \frac{\rho^*}{\rho_s}}$$

Closed-Cell Foam
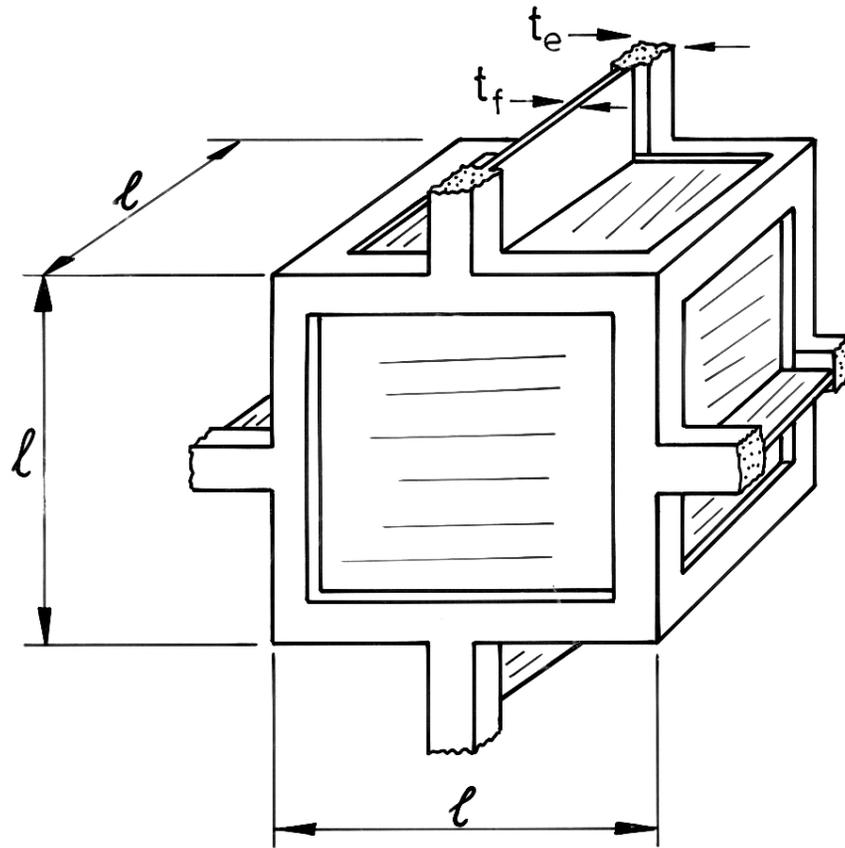
Cell Membrane Stretching
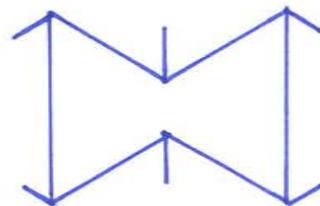

Gibson, L. J., and M. F. Ashby. *Cellular Solids: Structure and Properties*. 2nd ed. Cambridge University Press, © 1997. Figure courtesy of Lorna Gibson and Cambridge University Press.
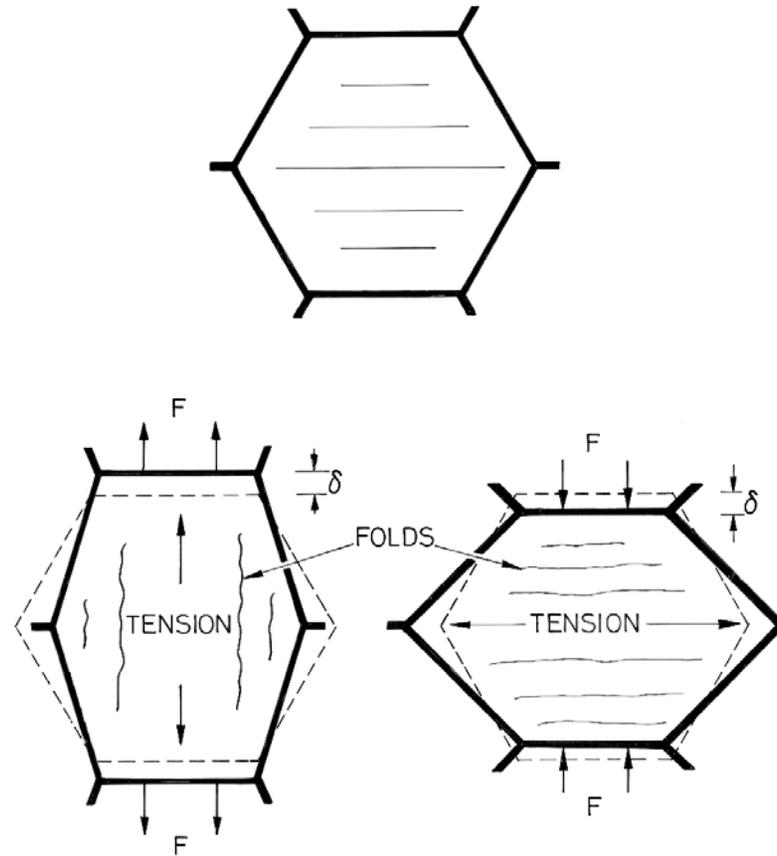
Closed cell foams – gas within cell may also contribute to E^*

- Cubic element of foam of volume V_o

- under uniaxial stress, axial strain in direction of stress is ϵ

- deformed volume V is:

$$\frac{V}{V_o} = 1 - \epsilon(1-2v^*)$$

taking compressive strain as positive
neglecting ϵ^2, ϵ^3 terms

$$\frac{V_g}{V_g^o} = \frac{1-\epsilon(1-2v^*) - \rho^*/\rho_s}{1-\rho^*/\rho_s}$$

V_g = volume gas

V_g^o = " " initially

- Boyle's law: $p V_g = p_o V_g^o$

p = pressure after strain ϵ
p_o = pressure initially

- Pressure that must be overcome is $p' = p - p_o$

$$p' = \frac{p_o \epsilon (1-2v^*)}{1 - \epsilon(1-2v^*) - \rho^*/\rho_s}$$

- contribution of gas compression to the modulus, E^*:

$$E_g^* = \frac{dp'}{d\epsilon} = \frac{p_o(1-2v^*)}{1-\rho^*/\rho_s}$$

ϵ 1

$V_0 = l_0^3$

$V = l_1 l_2 l_3$

$\epsilon_1 = \dfrac{l_1 - l_0}{l_0} \implies l_1 = l_0 + \epsilon_1 l_0 = l_0(1 + \epsilon_1)$

$\epsilon_2 = \dfrac{l_2 - l_0}{l_0} \implies l_2 = l_0 + \epsilon_2 l_0$

$\nu = -\dfrac{\epsilon_2}{\epsilon_1}$

$= l_0 - \nu \epsilon_1 l_0$

$\epsilon_2 = -\nu \epsilon_1$

$= l_0(1 - \nu \epsilon_1)$

$\epsilon_3 = l_0(1 - \nu \epsilon_1)$

$V = l_1 l_2 l_3 = l_0(1 + \epsilon_1)\, l_0(1 - \nu \epsilon_1)\, l_0(1 - \nu \epsilon_1) = l_0^3 (1 + \epsilon_1)(1 - \nu \epsilon)^2$

$\dfrac{V}{V_0} = \dfrac{l_0^3 (1 + \epsilon)(1 - \nu \epsilon)^2}{l_0^3} = (1 + \epsilon)(1 - 2\nu \epsilon + \nu^2 \epsilon^2)$

$= (1 - 2\nu \epsilon + \nu^2 \epsilon^2) + \epsilon - 2\nu \epsilon^2 + \nu^2 \epsilon^3$

$= 1 + \epsilon + 2\nu \epsilon$ (taking comp. as +)

$= 1 - \epsilon(1 - 2\nu)$

$$\rho' = \rho - \rho_0$$

$$\rho = \frac{\rho_0 V_g^0}{V_g}$$

$$\rho' = \rho - \rho_0 = \frac{\rho_0 V_g^0}{V_g} - \rho_0 = \rho_0 \left(\frac{V_g^0}{V_g} - 1 \right) \quad \cancel{\rho_0 \left(\frac{V_g^0 - V_g}{V_g} \right)}$$

$$= \rho_0 \left[\frac{1 - \rho/\rho_s}{1 - \epsilon(1 - 2\nu^*) - \rho^*/\rho_s} - 1 \right]$$

$$= \rho_0 \left[\frac{\cancel{1} - \rho/\rho_s \quad - \quad \left(\cancel{1} - \epsilon(1 - 2\nu^*) - \rho^*/\rho_s \right)}{1 - \epsilon(1 - 2\nu^*) - \rho^*/\rho_s} \right]$$

$$= \rho_0 \left[\frac{\epsilon(1 - 2\nu^*)}{1 - \epsilon(1 - 2\nu^*) - \rho^*/\rho_s} \right] \checkmark$$

Closed cell foam

$$\frac{E^*}{E_s} = \phi^2\left(\frac{\rho^*}{\rho_s}\right)^2 + (1-\phi)\left(\frac{\rho^*}{\rho_s}\right) + \frac{p_o(1-2\nu^*)}{E_s(1-\rho^*/\rho_s)}$$

$$\underset{\text{edge bending}}{\downarrow} \qquad \underset{\text{face stretching}}{\downarrow} \qquad \underset{\text{gas compression}}{\downarrow}$$

- note: if $p_o = p_{atm} = 0.1\,MPa$, gas compression term negligible,
  except for closed-cell elastomeric foams

- gas comp. can be significant if $p_o \gg p_{atm}$; also modifies shape of stress plateau
  in elastomeric closed cell foams

Shear modulus: edge bending, face stretching; shear $\Delta V = 0$ gas contrib. $= 0$

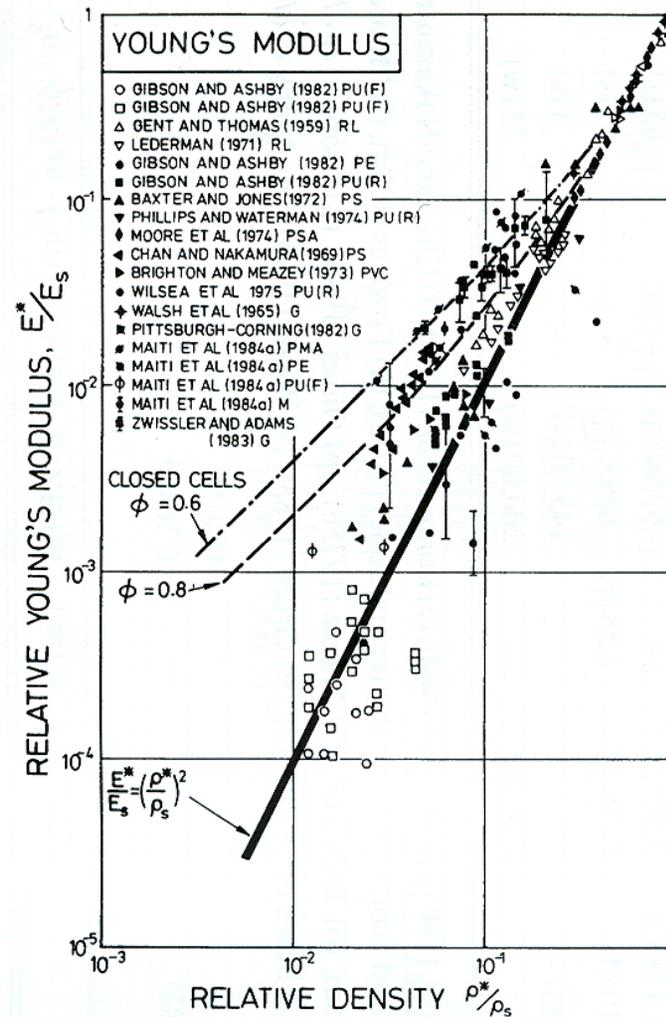$$\frac{G^*}{E_s} = \frac{3}{8}\left[\phi^2\left(\frac{\rho^*}{\rho_s}\right)^2 + (1-\phi)\left(\frac{\rho^*}{\rho_s}\right)\right] \qquad \text{(isotropic foam)}$$

Poisson's ratio = f (cell geometry only) $\quad \nu^* \approx 1/3$

Comparison with data

- data for polymers, glasses, elastomers
- E_s, ρ_s Table 5.1 in book
- open cells - open symbols
- closed cells - filled symbols

Young's Modulus

Shear Modulus
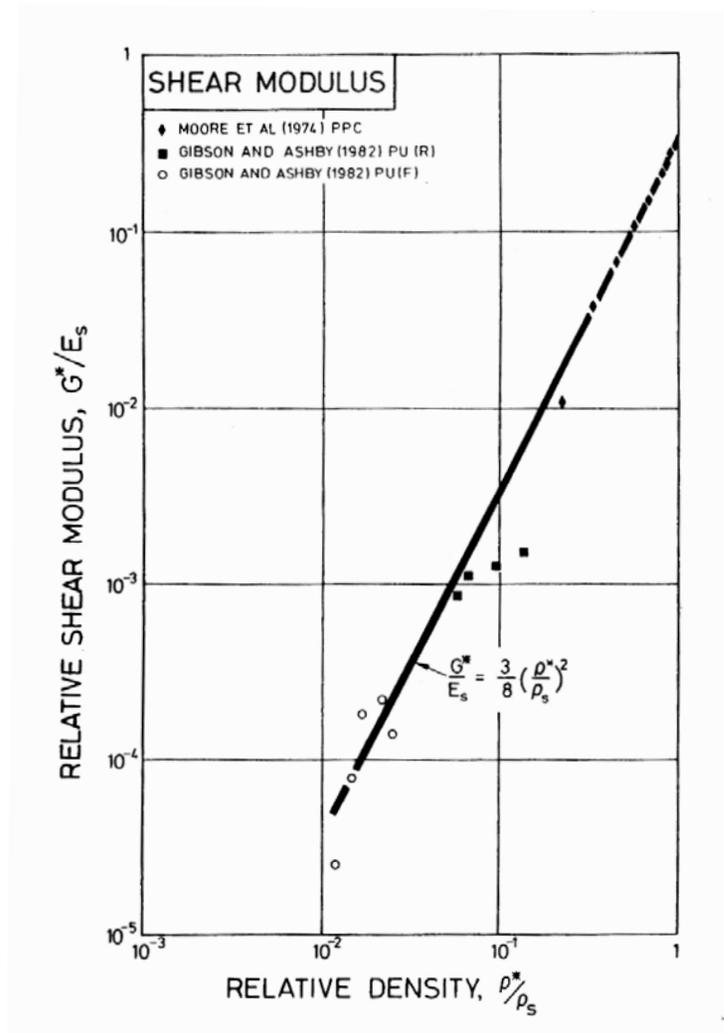
Gibson, L. J., and M. F. Ashby. *Cellular Solids: Structure and Properties*. 2nd ed. Cambridge University Press, © 1997. Figure courtesy of Lorna Gibson and Cambridge University Press.

Poisson's Ratio

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Non-linear elasticity

Open cells:

$$P_{cr} = \frac{n^2 \pi^2 E_s I}{\ell^2}$$

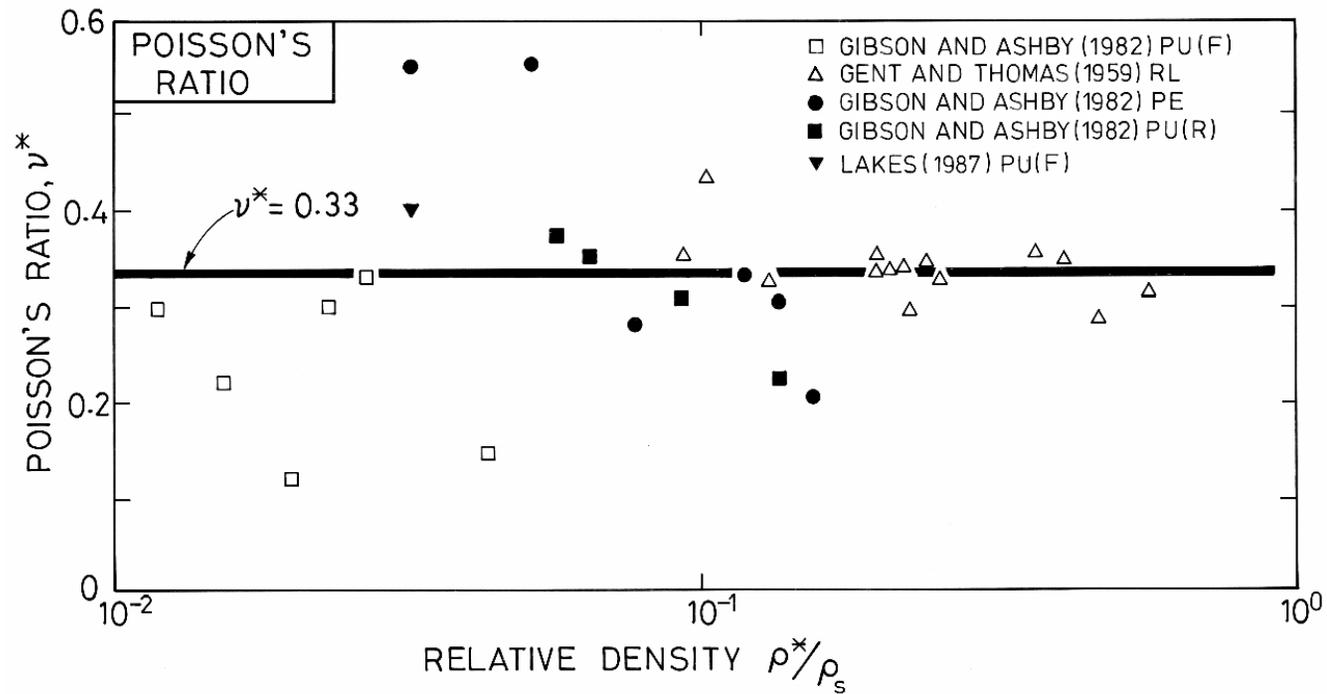$$\sigma^*_{el} \propto \frac{P_{cr}}{\ell^2} \propto E_s \left(\frac{t}{\ell}\right)^4$$

$$\boxed{\sigma^*_{el} = C_4 E_s (\rho^*/\rho_s)^2}$$

Data: $C_4 \approx 0.05$, corresponds to strain when buckling initiates, since

$$E^* = E_s (\rho^*/\rho_s)^2$$

Closed cells

- t_f often small compared to t_e (surface tension in processing) - contrib. small

- if $p_o \gg p_{atm}$, cell walls pretensioned; buckling stress has to overcome this

$$\sigma^*_{el} = 0.05 E_s (\rho^*/\rho_s)^2 + p_o - p_{atm}$$

- post-collapse behaviour - stress plateau rises due to gas compression (if faces don't rupture) $\nu^* = 0$ in post-collapse regime

$$p' = \frac{p_o \epsilon (1 - 2\nu^*)}{1 - \epsilon(1 - 2\nu^*) - \rho^*/\rho_s} = \frac{p_o \epsilon}{1 - \epsilon - \rho^*/\rho_s}$$

$$\boxed{\sigma^*_{post\text{-}collapse} = 0.05 E_s \left(\frac{\rho^*}{\rho_s}\right)^2 + \frac{p_o \epsilon}{1 - \epsilon - \rho^*/\rho_s}}$$
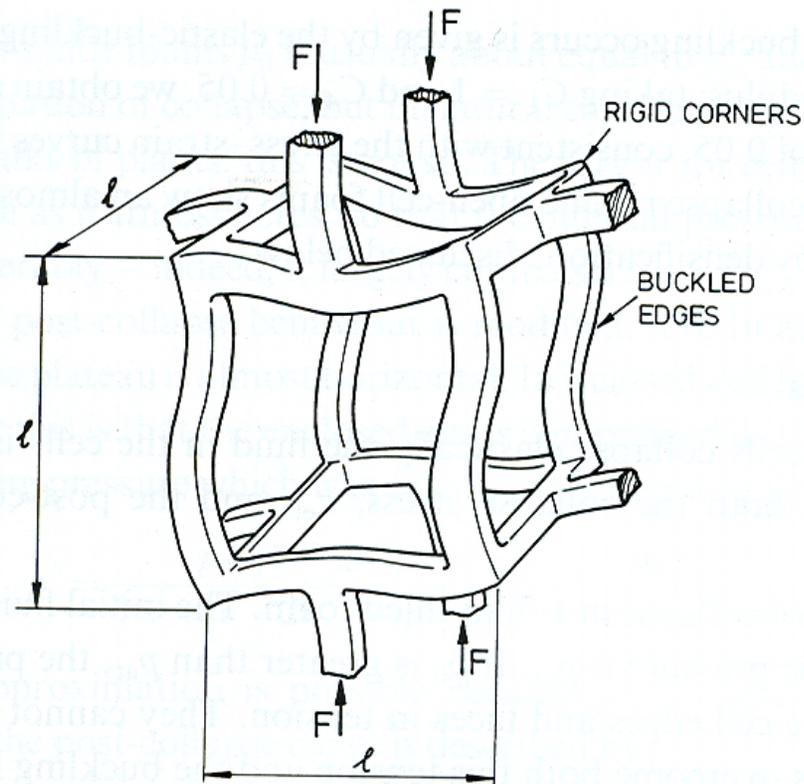
Elastic Collapse Stress

Elastic Collapse Stress
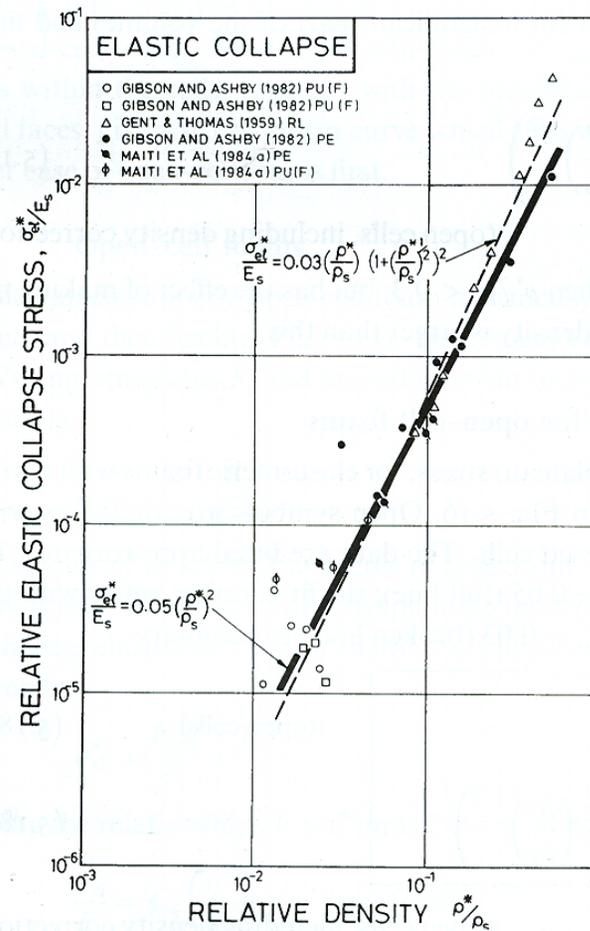
Gibson, L. J., and M. F. Ashby. *Cellular Solids: Structure and Properties*. 2nd ed. Cambridge University Press, © 1997. Figure courtesy of Lorna Gibson and Cambridge University Press.

Post-collapse stress strain curve

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Plastic collapse

Open cells

- failure when $M = M_p$

- $M_p \propto \sigma_{ys} t^3 \qquad M \propto \sigma_{pl}^* l^3$

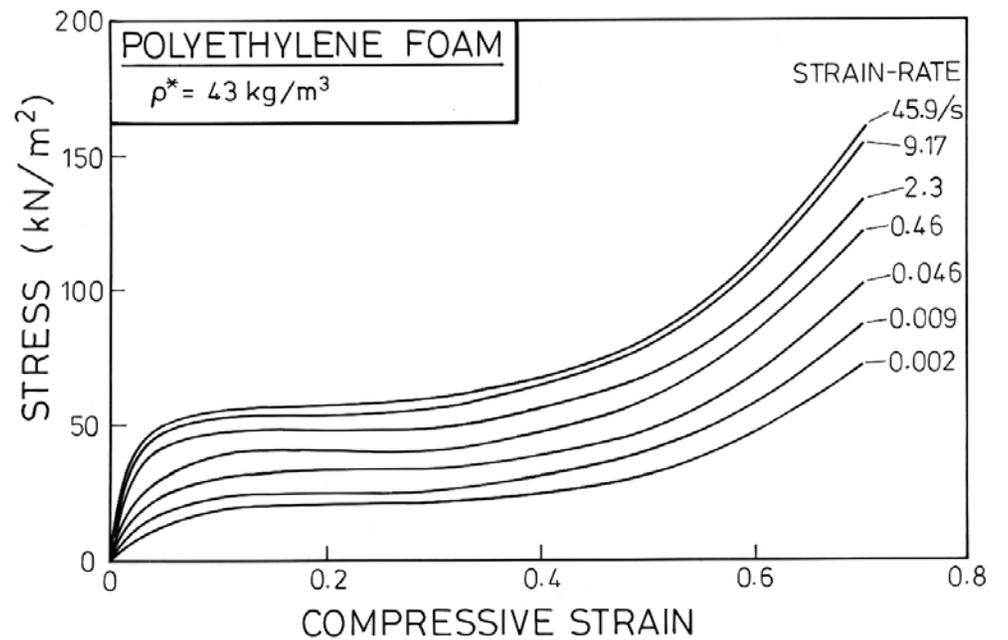$$\boxed{\sigma_{pl}^* = C_5 \, \sigma_{ys} \, (\rho^*/\rho_s)^{3/2}} \qquad C_5 \sim 0.3, \text{ from data.}$$

- elastic collapse precedes plastic collapse if $\sigma_{el}^* < \sigma_{pl}^*$

- $0.05 \, E_s \, (\rho^*/\rho_s)^2 \leq 0.3 \, \sigma_{ys} \, (\rho^*/\rho_s)^{3/2}$ rigid polymers $(\rho^*/\rho_s)_{cr} < 0.04 \quad \left(\frac{\sigma_{ys}}{E_s} \sim \frac{1}{30}\right)$

$\quad (\rho/\rho_s)_{critical} \leq 36 \, (\sigma_{ys}/E_s)^2$ metals $(\rho^*/\rho_s)_{cr} < 10^{-5} \quad \left(\frac{\sigma_{ys}}{E_s} \sim \frac{1}{1000}\right)$

Closed cells

- including all terms $\sigma_{pl}^* = C_5 \, \sigma_{ys} \, (\phi \, \rho^*/\rho_s)^{3/2} + C_5' \, \sigma_{ys} \, (1-\phi)\left(\frac{\rho^*}{\rho_s}\right) + p_i - p_{atm}$

$\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \qquad\qquad\qquad\qquad \uparrow \qquad\qquad\qquad \uparrow$

$\qquad\qquad\qquad\qquad\qquad\quad$ edge bending $\qquad$ face stretching $\qquad$ gas

- but in practice, faces often rupture around σ_{pl}^* - often $\sigma_{pl}^* = 0.3 \, (\rho^*/\rho_s)^{3/2} \, \sigma_{ys}$

Plastic Collapse Stress
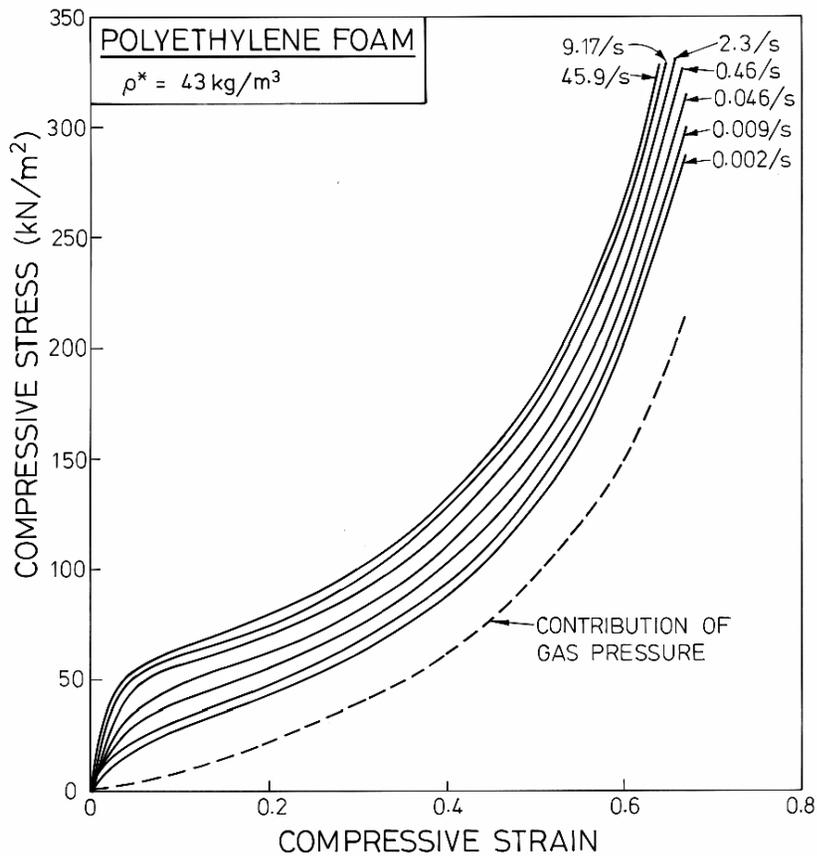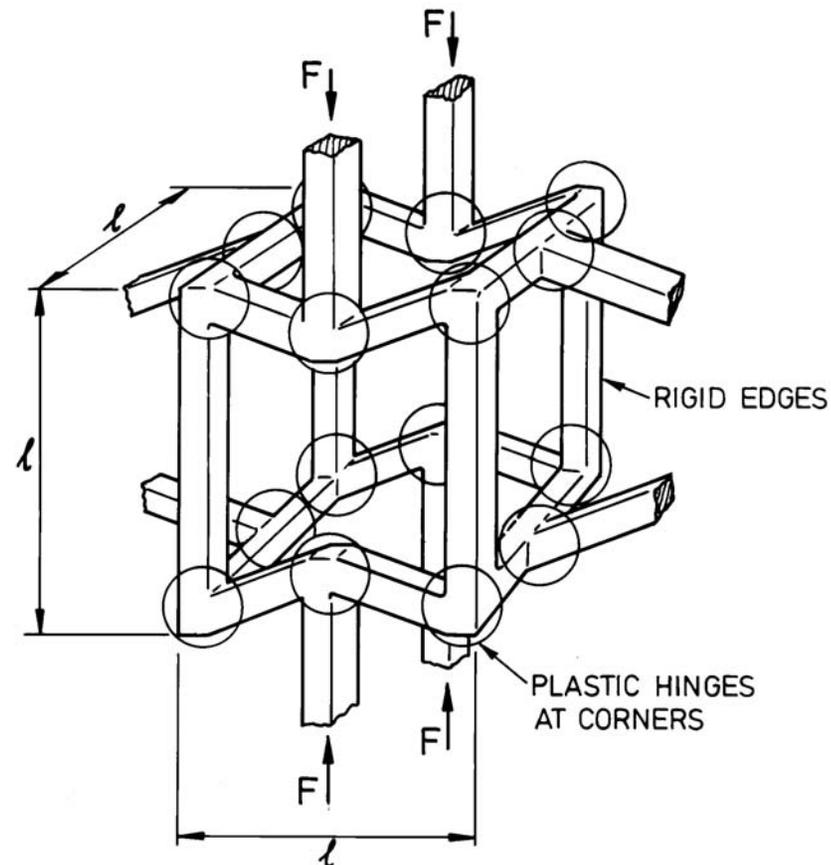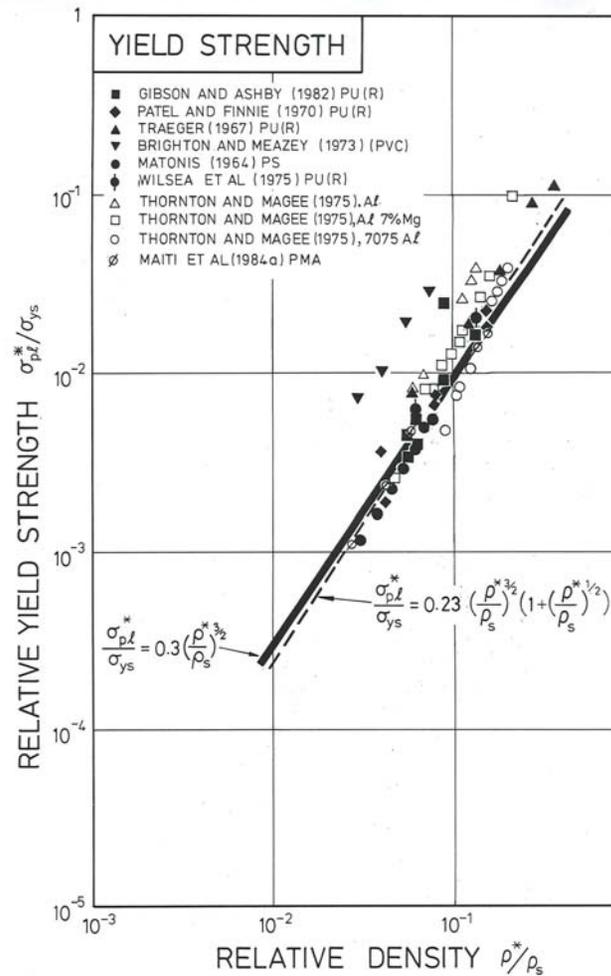
Plastic Collapse Stress

Gibson, L. J., and M. F. Ashby. *Cellular Solids: Structure and Properties*. 2nd ed. Cambridge University Press, © 1997. Figure courtesy of Lorna Gibson and Cambridge University Press.

Brittle crushing strength

Open cells

- failure when $M = M_f$ $M \alpha \, \sigma_{cr}^* \, l^3$ $M_f \propto \sigma_{fs} \, t^3$

$$\boxed{\sigma_{cr}^* = C_6 \, \sigma_{fs} \left(\rho^*/\rho_s\right)^{3/2}}$$ $C_6 \approx 0.2$

Densification strain, ϵ_D

- at large comp. strain, cell walls begin to touch, σ-ϵ rises steeply
- $E^* \rightarrow E_s$; σ-ϵ curve looks vertical, at limiting strain
- Might expect $\epsilon_D = 1 - \rho^*/\rho_s$
- Walls jam together at slightly smaller strain than this:

$$\boxed{\epsilon_D = 1 - 1.4 \, \rho^*/\rho_s}$$

Densification Strain
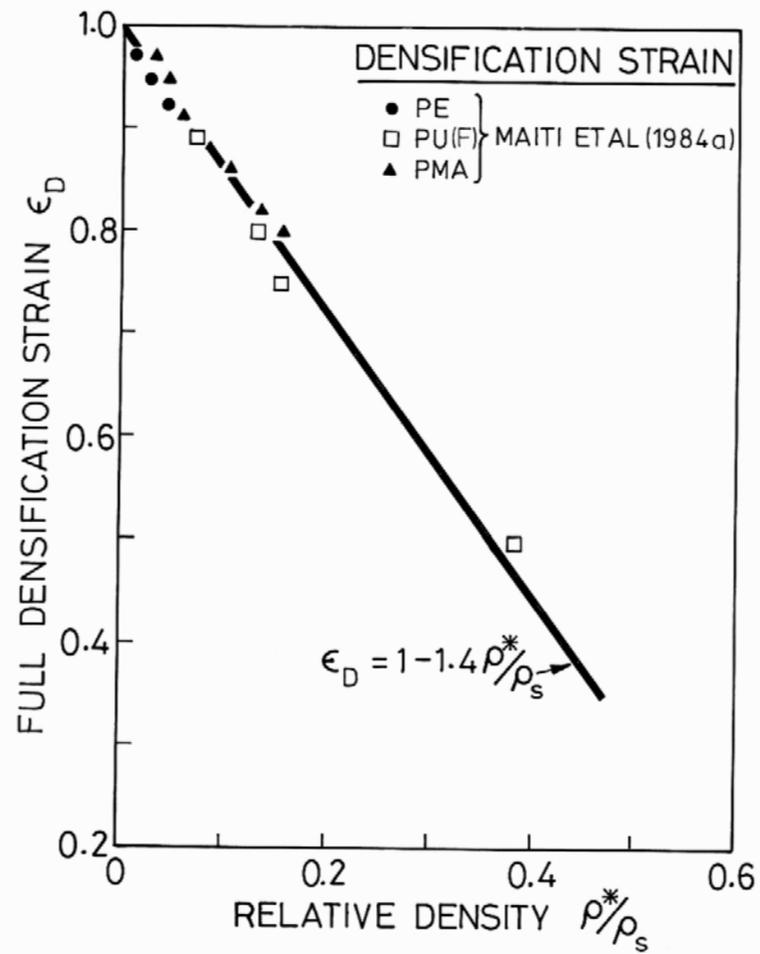
Gibson, L. J., and M. F. Ashby. *Cellular Solids: Structure and Properties*. 2nd ed. Cambridge University Press, © 1997. Figure courtesy of Lorna Gibson and Cambridge University Press.

Fracture toughness

Open cells : crack length $2a$, local stress σ_l, remote stress σ^∞

$$\sigma_l = \frac{C \sigma^\infty \sqrt{\pi a}}{\sqrt{2\pi r}} \qquad \text{a distance } r \text{ from crack tip}$$

· next unbroken cell wall a distance $r \approx l/2$ ahead of crack tip subject to a force (integrating stress over next cell)

$$F \propto \sigma_l \, l^2 \propto \sigma^\infty \sqrt{\frac{a}{l}} \, l^2$$

· edges fail when applied moment, $M =$ fracture moment, M_f

$$M_f \propto \sigma_{fs} \, t^3$$

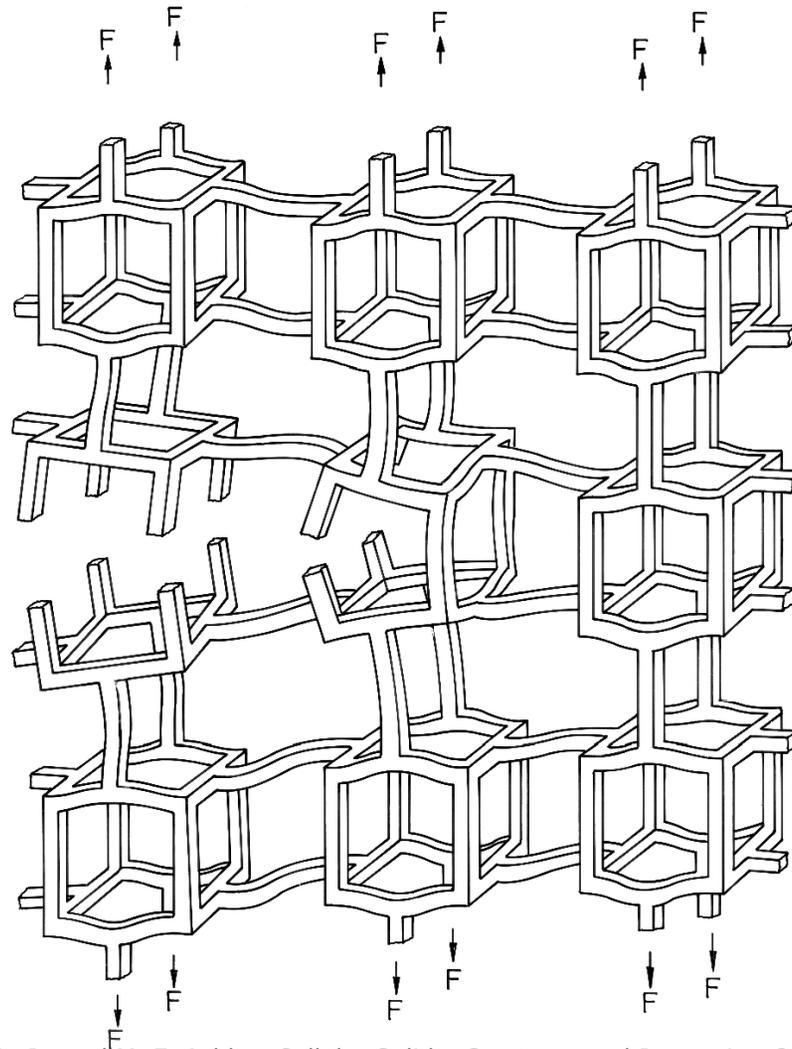$$M \propto F l \propto \sigma^\infty \left(\frac{a}{l}\right)^{1/2} l^3 \qquad M = M_f \implies \sigma^\infty \left(\frac{a}{l}\right)^{1/2} l^3 \propto \sigma_{fs} \, t^3$$

$$\sigma^\infty \propto \sigma_{fs} \left(\frac{l}{a}\right)^{1/2} \left(\frac{t}{l}\right)^3$$

$$\boxed{K_{IC}^* = \sigma^\infty \sqrt{\pi a} = C_8 \, \sigma_{fs} \sqrt{\pi l} \, (\rho^*/\rho_s)^{3/2}}$$

Data: $C_8 \sim 0.65$

Fracture Toughness

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Fracture Toughness

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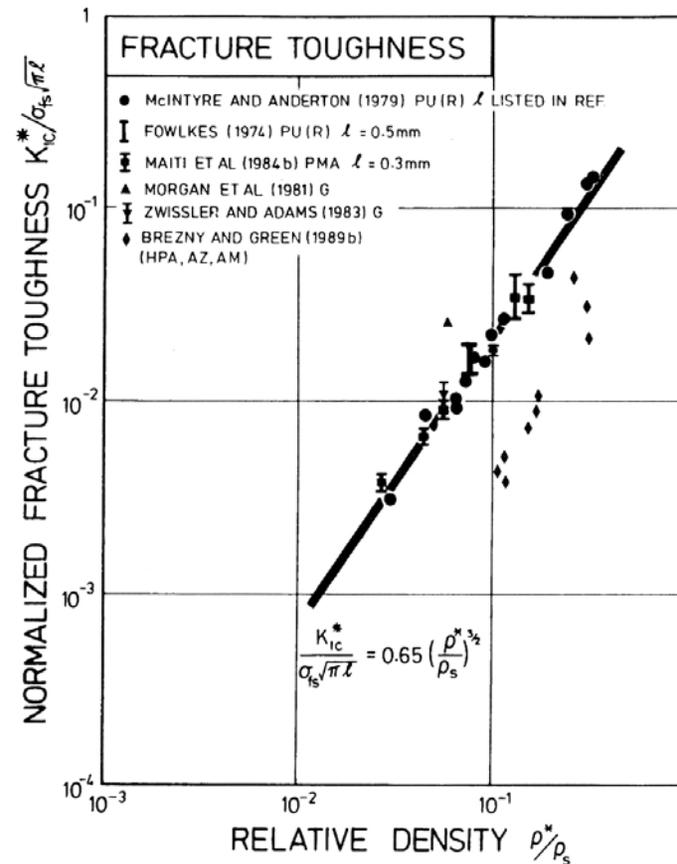
3.054 / 3.36 Cellular Solids: Structure, Properties and Applications
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