18.01 Exam 2

Problem 1(20 points) Compute the following derivatives. Show all work, or you will not receive credit.

(a)(10 points)

$$\frac{d}{d\theta}\left(\frac{2\tan(\theta)}{1-(\tan(\theta))^2}\right)$$

Solution to (a) The simplest solution uses the double-angle formula for $\tan(\theta)$. Because $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ and $\cos(2\theta) = \cos(\theta)^2 - \sin(\theta)^2$, $\tan(2\theta)$ equals,

$$\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)} = \frac{2\sin(\theta)\cos(\theta)}{\cos(\theta)^2 - \sin(\theta)^2}.$$

Factoring $\cos(\theta)^2$ from both numerator and denominator yields,

$$\tan(2\theta) = \frac{2\sin(\theta)/\cos(\theta)}{1-(\sin(\theta)/\cos(\theta))^2} = \frac{2\tan(\theta)}{1-\tan(\theta)^2}.$$

This gives,

$$\frac{d}{d\theta}\left(\frac{2\tan(\theta)}{1-\tan(\theta)^2}\right) = \frac{d}{d\theta}\tan(2\theta).$$

Let u be 2θ. By the chain rule,

$$\frac{d}{d\theta}\tan(u) = \frac{d\tan(u)}{du}\frac{du}{d\theta} = \sec(u)^2 2.$$

Therefore,

$$\frac{d}{d\theta}\left(\frac{2\tan(\theta)}{1-\tan(\theta)^2}\right) = \boxed{2\sec(2\theta)^2.}$$

The more straightforward proof uses the chain rule and the quotient rule. Set $v = \tan(\theta)$. Set $w = 2v/(1-v^2)$. By the chain rule,

$$\frac{dw}{d\theta} = \frac{dw}{dv}\frac{dv}{d\theta}.$$

By the quotient rule,

$$\frac{dw}{dv} = \frac{2}{(1-v^2)^2}((2v)'(1-v^2) - (2v)(1-v^2)') = \frac{2}{(1-v^2)^2}(2(1-v^2) - 2v(-2v)) = \frac{4(1+v^2)}{(1-v^2)^2}.$$

Also,

$$\frac{d}{d\theta}\tan(\theta) = \sec(\theta)^2.$$

By the chain rule,

$$\frac{dw}{d\theta} = \frac{4(1+v^2)}{(1-v^2)^2}\sec(\theta)^2 = \frac{4(1+\tan(\theta)^2)\sec(\theta)^2}{(1-\tan(\theta)^2)^2}.$$

Since $1+\tan(\theta)^2$ equals $\sin(\theta)^2$, this simplifies to,

$$\frac{d}{dw}\left(\frac{2\tan(\theta)}{1-\tan(\theta)^2}\right) = \boxed{4\sec(\theta)^4/(1-\tan(\theta)^2)^2.}$$

(b)(10 points)

$$\frac{d}{dt}\sqrt{1-(\sin(2t))^2}, \quad -\pi/4 < t < \pi/4.$$

Solution to (b) By the Pythagorean theorem, $\sin(2t)^2 + \cos(2t)^2$ equals 1. This gives,

$$\sqrt{1-\sin(2t)^2} = \cos(2t).$$

There is an issue about whether this is $+\cos(2t)$ or $-\cos(2t)$. The condition that t is between $-\pi/4$ and $\pi/4$ insures it is $+\cos(2t)$. Let u equal $2t$ and let v equal $\cos(u)$. By the chain rule,

$$\frac{dv}{dt} = \frac{dv}{du}\frac{du}{dt}.$$

Of course,

$$\frac{d\cos(u)}{du} = -\sin(u), \text{ and } \frac{d(2t)}{dt} = 2.$$

Plugging in gives,

$$\frac{d}{dt}\sqrt{1-\sin(2t)^2} = \boxed{-2\sin(2t).}$$

Problem 2(40 points) For $x > 0$, the function $f(x)$ is defined by,

$$f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}.$$

For the purposes of this problem, $\sqrt{2} \approx 1.4$, $\sqrt{3} \approx 1.7$ and $\sqrt{5} \approx 2.2$. Be sure you work with the correct function. If you work with the wrong function, few points will be given.

(a)(3 points) Write the equation of each vertical asymptote. If none exist, write "none exist".

Solution to (a) For every vertical asymptote $x = a$, the denominator of some term in the expression approaches 0 as x approaches a. The only denominator in the expression is $\sqrt{x}$, which is zero only when $x = 0$. Thus there is a unique vertical asymptote:

$$\boxed{\text{the line } x = 0.}$$

(b)(3 points) Write the equation of each horizontal asymptote. If none exist, write "none exist".

Solution to (b) As x approaches $+\infty$, $f(x)$ grows like $\sqrt{x}$. Thus,

$$\boxed{\text{there is no horizontal asymptote.}}$$

(c)(7 points) On the number line, identify where $f'(x)$ is positive, negative or zero.

Solution to (c) The expression $f(x)$ equals,

$$f(x) = x^{-1/2}(x+1).$$

By the product rule,

$$\frac{df}{dx} = \frac{dx^{-1/2}}{dx}(x+1) + x^{-1/2}\frac{d(x+1)}{dx} = \frac{-1}{2}x^{-3/2}(x+1) + x^{-1/2} = \frac{x^{-3/2}}{2}(-(x+1)+2x).$$

Simplifying gives,

$$f'(x) = \frac{x^{-3/2}}{2}(x-1).$$

This is positive, negative or zero as $x - 1$ is positive, negative or zero. Thus,

$$\boxed{f(x) \text{ is positive for } x > 1, \text{ it is negative for } 0 < x < 1, \text{ and it is zero for } x = 1.}$$

(d)(3 points) Write the coordinates of each local maximum. If none exist, write "none exist".

Solution to (d) By part (c), $f(x)$ is decreasing for $0 < x < 1$ and is increasing for $x > 1$. Therefore,

$$\boxed{\text{there is no local maximum.}}$$

(e)(3 points) Write the coordinates of each local minimum. If none exist, write "none exist".

Solution to (d) By the same argument as in (d),

$$\boxed{\text{there is a unique local minimum at } x = 1.}$$

(f)(4 points) Write the coordinates of each inflection point. If none exist, write "none exist".

Solution to (f) By the product rule, the second derivative of $f(x)$ is,

$$\frac{d}{dx}\left(\frac{x^{-3/2}}{2}(x-1)\right) = \frac{1}{2}\frac{d(x^{-3/2})}{dx}(x-1) + \frac{1}{2}x^{-3/2}\frac{d(x-1)}{dx} = \frac{-3}{4}x^{-5/2}(x-1) + \frac{1}{2}x^{-3/2}.$$

This simplifies to,

$$\frac{d^2 f}{dx^2} = \frac{x^{-5/2}}{4}(-x+3).$$

This is positive for $0 < x < 3$, negative for $x > 3$ and zero for $x = 3$. Notice $f(3) = \sqrt{3} + 1/\sqrt{3} = \frac{4\sqrt{3}}{3}$. Therefore,

$$\boxed{\text{there is a unique inflection point with coordinates } (3, 4\sqrt{3}/3).}$$

(g)(7 points) On the number line, identify where $f''(x)$ is positive, negative or zero.

Solution to (g) By the previous part,

$$\boxed{f''(x) \text{ is positive for } 0 < x < 3, \text{ negative for } x > 3 \text{ and zero for } x = 3.}$$

(h)(10 points) On the grid given, sketch the graph of $y = f(x)$.

Solution to (h) The sketch of the graph is given in Figure 1. The local minimum is the green dot, and the inflection point is the red star.

Problem 3(30 points) A box is made from two identical square sheets of metal with edge length E. A small square of edge length x is removed and the two flaps are folded up. Find the value of x that maximizes the volume of the box.

Solution to Problem 3 The length and width of the base of the box are each $l = E - x$ and $w = E - x$. The height of the box is $h = x$. The volume of the box is,

$$V = lwh = (E-x)^2 x.$$

Therefore,

$$\frac{dV}{dx} = 2(E-x)(-1)x + (E-x)^2 = (E-x)(-2x + (E-x)) = (E-x)(E-3x).$$

The critical points are $x = E/3$ and $x = E$. The endpoints are $x = 0$ and $x = E$. Since $V(0) = 0$ and $V(E) = 0$, the maximum volume occurs when $x = E/3$, giving

$$\boxed{V = 4E^3/27 \text{ when } x = E/3.}$$

Problem 4(10 points) Find the quadratic approximation of $\sqrt{2 - \cos(\theta)}$ for $\theta \approx 0$.

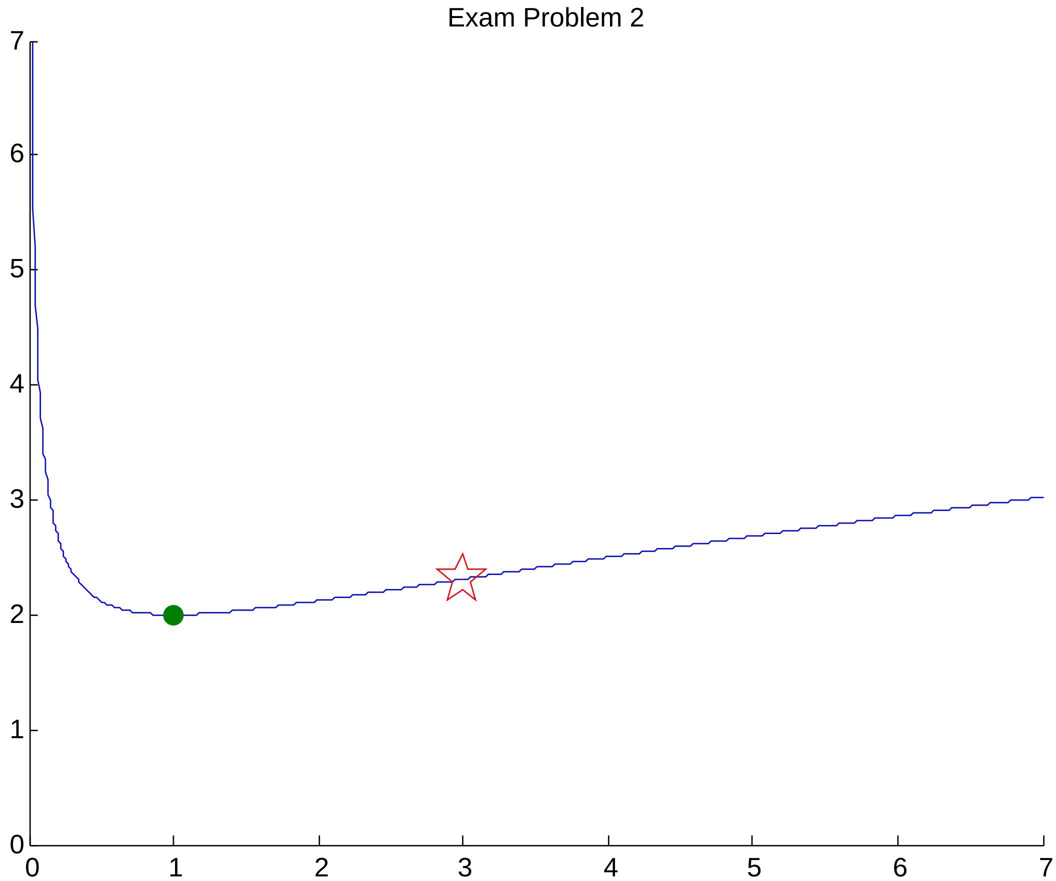

Figure 1: Graph of the function

Solution to Problem 4 The quadratic approximation of $\cos(\theta)$ at $\theta \approx 0$ is,

$$\cos(\theta) \approx 1 - \frac{\theta^2}{2} \text{ for } \theta \approx 0.$$

Thus, the quadratic approximation of the inner expression is,

$$2 - \cos(\theta) \approx 1 + \frac{\theta^2}{2} \text{ for } \theta \approx 0.$$

The quadratic approximation for $\sqrt{1+x}$ for $x \approx 0$ is,

$$\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{4} \text{ for } x \approx 0.$$

Substituting $\theta^2/2$ for x gives the quadratic approximation of $\sqrt{2-\cos(\theta)}$,

$$\sqrt{2-\cos(\theta)} \approx 1 + \frac{1}{2}\left(\frac{\theta^2}{2}\right) + \frac{-1}{4}\left(\frac{\theta^2}{2}\right)^2 \approx \boxed{1 + \theta^2/4 \text{ for } \theta \approx 0}.$$