Lecture 28. December 1, 2005

Homework. Problem Set 8 Part I: (a) and (b).

Practice Problems. Course Reader: 6A-1, 6A-2.

1. Indeterminate forms. Expressions of the form $0/0$, ∞/∞, $0 \times \infty$, $\infty - \infty$, 0^∞ and ∞^0 are called *indeterminate forms*. To be precise, none of these expressions is defined in mathematics. However, if a naive limit computation $\lim_{x \to a} F(x)$ leads to an indeterminate form, it often happens that a more careful computation using calculus eliminates the indeterminate form.

Example. Let b be any real number. Compute the limit as x approaches 0 of $F(x) = (b+1/x) - 1/x$, $x \neq 0$. If we evaluate this limit in a naive manner, we get,

$$\lim_{x \to 0} F(x) = \lim_{x \to 0} \left(b + \frac{1}{x}\right) - \left(\frac{1}{x}\right) \text{ ``$=$'' } \lim_{x \to 0} b + \frac{1}{x} - \lim_{x \to 0} \frac{1}{x} = \infty - \infty.$$

This is an indeterminate form. In other words, the computation of the limit failed to give any useful information. The reason is that the general formula,

$$\lim_{x \to a} [g(x) + h(x)] = \lim_{x \to a} g(x) - \lim_{x \to a} h(x),$$

only holds if all three limits are defined, which they are not in our case.

Of course $F(x)$ is simply the constant function with value b. Therefore,

$$\lim_{x \to 0} F(x) = \lim_{x \to 0} b = \boxed{b.}$$

Thus, a more careful computation proves the limit exists and gives its value.

2. The Mean Value Theorem revisited. Recall the Mean Value Theorem: If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then for some c strictly between a and b,

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Thus, given two such functions $f(x)$ and $g(x)$ such that $g(b) - g(a)$ is nonzero, there exist two values c_1 and c_2 strictly between a and b such that,

$$\frac{f'(c_1)}{g'(c_2)} = \frac{(f(b)-f(a))/(b-a)}{(g(b)-g(a))/(b-a)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Is there a single value $c = c_1 = c_2$ where this equality holds?

The answer is yes. Form the function

$$F(x) = (f(b)-f(a))(g(x)-g(a)) - (g(b)-g(a))(f(x)-f(a)).$$

Since $f(x)$ and $g(x)$ are continuous on $[a,b]$, also $F(x)$ is continuous on $[a,b]$. Since $f(x)$ and $g(x)$ are differentiable on (a,b), also $F(x)$ is differentiable on (a,b). Moreover,

$$F(a) = F(b) = 0.$$

Thus, by the Mean Value Theorem, there exists a value c strictly between a and b such that $F'(c) = 0$. By a straightforward computation,

$$F'(c) = (f(b)-f(a))g'(c) - (g(b)-g(a))f'(c).$$

This proves the *Generalized Mean Value Theorem*. The main consequence of the Generalized Mean Value Theorem is the following result.

Proposition. Let $f(x)$ and $g(x)$ be continuous functions on $[a,b]$ that are differentiable on (a,b). If $g'(x)$ is nonzero on (a,b), then $g(x)-g(a)$ is nonzero for all $a < x < b$ so that the expression,

$$\frac{f(x)-f(a)}{g(x)-g(a)}$$

is defined. The right-handed limit,

$$\lim_{x\to a^+} \frac{f(x)-f(a)}{g(x)-g(a)},$$

exists if and only if the right-handed limit,

$$\lim_{x\to a^+} \frac{f'(x)}{g'(x)},$$

exists. If both limits exist, they are equal,

$$\lim_{x\to a^+} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

A similar result holds for left-handed limits. The proof follows by applying the Generalized Mean Value Theorem to the interval $[a,x]$ to replace $(f(x)-f(a))/(g(x)-g(a))$ by $f'(c)/g'(c)$. Then x approaches a as c approaches a.

3. L'Hospital's rule. The most important case of the proposition is *L'Hospital's rule.* This is exactly the case when $f(a) = g(a) = 0$. In this case, a naive computation would give,

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} \text{ "="} \frac{f(a)}{g(a)} = \frac{0}{0},$$

which is an indeterminate form. Again, the problem is that the general formula,

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a^+} f(x)}{\lim_{x \to a^+} g(x)},$$

only holds if all three limits are defined, and the limit $\lim_{x \to a^+} g(x)$ is nonzero. Since the limit is zero, the formula does not hold.

However, if $f'(x)$ and $g'(x)$ exist, and if $g'(x)$ is nonzero, then the proposition has the following consequence, known as L'Hospital's rule,

$$\boxed{\lim_{x \to a^+} f(x)/g(x) = \lim_{x \to a^+} f'(x)/g'(x).}$$

Examples.

$$\lim_{x \to 0} \frac{\sinh(x)}{\sin(x)} = \lim_{x \to 0} \frac{\cosh(x)}{\cos(x)} = \frac{1}{1} = \boxed{1.}$$

$$\lim_{x \to 2} \frac{4x^3 - 32}{x^2 - x - 2} = \lim_{x \to 2} \frac{12x^2}{2x - 1} = \frac{12 \cdot 4}{2 \cdot 2 - 1} = \frac{48}{3} = \boxed{16.}$$

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \lim_{x \to 0} \sin(x) 2x = \lim_{x \to 0} \cos(x) 2 = \boxed{1/2.}$$

4. L'Hospital's rule for other indeterminate forms. L'Hospital's rule can be used to compute limits that naively lead to indeterminate forms other than $0/0$. For instance, if

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = \infty,$$

then the naive computation gives,

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} \text{ "="} \frac{\infty}{\infty}.$$

Now observe that,

$$\lim_{x \to a^+} (1/f(x)) = \lim_{x \to a^+} (1/g(x)) = 0.$$

Therefore, if both $g(x)$ and $g'(x)$ are nonzero on (a, b), then L'Hospital's rule gives,

$$\lim_{x \to a^+} \frac{(1/f(x))}{(1/g(x))} = \lim_{x \to a^+} \frac{(1/f(x))'}{(1/g(x))'} = \lim_{x \to a^+} \frac{-f'(x)/f(x)^2}{-g'(x)/g(x)^2}.$$

Assuming that the limits,

$$\lim_{x\to a^+} \frac{f(x)}{g(x)}, \text{ and } \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$$

are defined and nonzero, the formula above can be re-written as,

$$\left(\lim_{x\to a^+} \frac{f(x)}{g(x)}\right)^{-1} = \left(\lim_{x\to a^+} \frac{f'(x)}{g'(x)}\right) \cdot \left(\lim_{x\to a^+} \frac{f(x)}{g(x)}\right)^{-2}.$$

Solving gives,

$$\lim_{x\to a^+} f(x)/g(x) = \lim_{x\to a^+} f'(x)/g'(x),$$

if both limits are defined and nonzero. In fact, a better result is true (with a more subtle proof): if the second limit is defined, then the first limit is defined and the 2 are equal (whether or not they are zero).

Example.

$$\lim_{x\to\pi/2^+} \frac{\ln(x-\pi/2)}{\sec(x)} = \lim_{x\to\pi/2^+} \frac{1/(x-\pi/2)}{\sec(x)\tan(x)} = \cdots = 0.$$

By similar arguments, other indeterminate forms can also be reduced to L'Hospital's rule. Also, limits of the form,

$$\lim_{x\to\infty} F(x)$$

giving indeterminate forms can often be reduced to L'Hospital's rule. The moral is that the formula,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

is almost always true if $f(a)/g(a)$ is an indeterminate form. But a certain amount of care should be used, since occasionally this fails.