Properties of integrals

In this section, we prove the four basic properties of the integral that we shall need.

Theorem. (Properties of the integral)

(1) (Linearity property.) If f and g are integrable on $[a,b]$, then so is $cf + dg$ (here c and d are constants), and furthermore

$$\int_a^b (cf+dg) = c \int_a^b f + d \int_a^b g.$$

(2) (Additivity property.) Suppose f is defined on $[a,c]$ and $a < b < c$. Then

$$\int_a^c f = \int_a^b f + \int_b^c f;$$

the two integrals on the right exist if and only if the integral on the left exists.

(3) (Comparison property.) If $f(x) \leq g(x)$ for all x in $[a,b]$, then

$$\int_a^b f \leq \int_a^b g,$$

provided both integrals exist.

(4) (Reflection property.) If f is integrable on $[a,b]$, then $f(-x)$ is integrable on $[-b,-a]$, and

$$\int_{-b}^{-a} f(-x)\, dx = \int_a^b f(x)\, dx.$$

We use the first three of these properties repeatedly. Property (4) is used only in deriving the formula for $\int_a^b x^p\,dx$.

Let us note that once we make the convention that

$$\int_a^a f = 0 \quad \text{and that} \quad \int_b^a f = -\int_a^b f \quad \text{if} \quad a < b,$$

then the formula

$$\int_a^c f = \int_a^b f + \int_b^c f$$

holds without regard to the requirement that $a < b < c$. The proof is left as an exercise.

<u>Proof.</u> First, one verifies these properties for step functions. This is quite straightforward. Property (3) has already been proved; properties (1) and (2) will be assigned as exercises; and property (4) is proved as follows:

Let s be a step function on $[a,b]$ relative to the partition $x_o,\ldots,x_n$. Let $s(x) = s_k$ for x in (x_{k-1},x_k). The function

$$u(x) = s(-x)$$

is then a step function relative to the partition $-x_k,\ldots,-x_1,-x_o$ of the interval $[-b,-a]$. Indeed, if x is in the interval $(-x_k,-x_{k-1})$, then $-x$ is in the interval (x_{k-1},x_k), so that

$$u(x) = s(-x) = s_k.$$

Then by definition,

$$\int_{-b}^{-a} u(x)\ dx = \sum_{k=n}^{1} s_k \cdot ((-x_{k-1}) - (-x_k)).$$

But

$$\int_a^b s(x)\ dx = \sum_{i=1}^{n} s_k \cdot (x_k - x_{k-1});$$

and these two expressions are equal. Thus (4) holds for step functions.

Step 2. We first prove property (1) in the case where c and d are non-negative. Suppose that f and g are integrable on $[a,b]$. Choose step functions s_i and t_i such that

$$s_1 \leq f \leq t_1 \quad \text{and} \quad s_2 \leq g \leq t_2$$

and

$$\int_a^b t_1 - \int_a^b s_1 < \frac{\varepsilon}{2(c+1)} \quad \text{and} \quad \int_a^b t_2 - \int_a^b s_2 < \frac{\varepsilon}{2(d+1)}.$$

Then let $s = cs_1 + ds_2$ and let $t = ct_1 + dt_2$. Now s and t are step functions, and (since c and d are non-negative)

$$s \leq cf + dg \leq t .$$

Furthermore, by property (1) for step functions,

$$\int_a^b t - \int_a^b s = [\, c\int_a^b t_1 + d\int_a^b t_2 \,] - [\, c\int_a^b s_1 + d\int_a^b s_2 \,]$$

$$= c\,[\int_a^b t_1 - \int_a^b s_1 \,] + d\,[\int_a^b t_2 - \int_a^b s_2 \,]$$

$$\leq \frac{c\varepsilon}{2(c+1)} + \frac{d\varepsilon}{2(d+1)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

Hence the integral of $cf + dg$ exists by the Riemann condition.

Now, by definition of the integral, we have

$$\int_a^b s_1 \leq \int_a^b f \leq \int_a^b t_1 \quad \text{and} \quad \int_a^b s_2 \leq \int_a^b g \leq \int_a^b t_2 .$$

We multiply the first set of inequalities by c, and the second by d, and add, obtaining the inequalities:

$$\int_a^b s = c\int_a^b s_1 + d\int_a^b s_2 \leq \boxed{c\int_a^b f + d\int_a^b g} \leq c\int_a^b t_1 + d\int_a^b t_2 = \int_a^b t .$$

Here we use property (1) for step functions again. Since the expression in the box lies between the integrals of s and t, by the Riemann condition it must equal the integral of $cf + dg$.

<u>Step 3</u>. To complete the proof of property (1), it suffices to show that

$$\int_a^b (-f) = -\int_a^b f.$$

This is easy. Given $\varepsilon > 0$, choose step functions s and t such that $s \leq f \leq t$ on $[a,b]$, and

$$\int_a^b t - \int_a^b s < \varepsilon \quad .$$

Then $-s$ and $-t$ are step functions on $[a,b]$, and $-t \leq -f \leq -s$ on $[a,b]$. Furthermore,

$$\left[\int_a^b -s\right] - \left[\int_a^b -t\right] = -\int_a^b s + \int_a^b t < \varepsilon.$$

Here we use property (1) for step functions. Thus the integral of $-f$ exists, by the Riemann condition.

Now by definition of the integral

$$\int_a^b s \leq \int_a^b f \leq \int_a^b t.$$

Multiplying these inequalities by -1, we conclude that

$$\int_a^b (-t) = -\int_a^b t \leq \boxed{-\int_a^b f} \leq -\int_a^b s = \int_a^b (-s) \ .$$

Here we use property (1) for step functions, again. Since the expression in the box lies between the integrals of $-t$ and $-s$, by the Riemann condition it must equal the integral of $-f$.

<u>Step 4.</u> Now we prove property (2). We consider first the "existence" part of the statement. Suppose the integrals

$$\int_a^b f \quad \text{and} \quad \int_b^c f$$

exist. Choose step functions s_1 and t_1 with $s_1 \leq f \leq t_1$ on $[a,b]$, and choose step functions s_2 and t_2 with $s_2 \leq f \leq t_2$ on $[b,c]$, such that

$$\int_a^b t_1 - \int_a^b s_1 < \varepsilon/2 \quad \text{and} \quad \int_b^c t_2 - \int_b^c s_2 < \varepsilon/2.$$

The values of these functions at the partition points do not matter, so we can assume that t_1 and t_2 are equal at c, and s_1 and s_2 are equal at c. Then t_1 and t_2 combine to define a step function t such that $f \leqslant t$ on $[a,c]$, and s_1 and s_2 combine to form a step function s such that $s \leqslant f$ on $[a,c]$. Furthermore, using property (2) for step functions,

$$\int_a^c t - \int_a^c s = \left(\int_a^b t + \int_b^c t\right) - \left(\int_a^b s + \int_b^c s\right)$$

$$= \left(\int_a^b t_1 + \int_b^c t_2\right) - \left(\int_a^b s_1 + \int_b^c s_2\right)$$

(by the way s and t were constructed)

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

Hence $\int_a^c f$ exists, by the Riemann condition.

Conversely, suppose $\int_a^c f$ exists. Then given $\varepsilon > 0$, we can choose step functions s and t with $s \leqslant f \leqslant t$ on $[a,c]$, such that

$$\int_a^c t - \int_a^c s < \varepsilon.$$

Let s_1 and t_1 be the restrictions of s and t, respectively, to $[a,b]$, and let s_2 and t_2 be their restrictions to $[b,c]$. As before, using property (2) for step fucntions, we have

$$\left(\int_a^b t_1 + \int_b^c t_2\right) - \left(\int_a^b s_1 + \int_b^c s_2\right) < \varepsilon,$$

or

$$\left(\int_a^b t_1 - \int_a^b s_1\right) + \left(\int_b^c t_2 - \int_b^c s_2\right) < \varepsilon.$$

Since each expression in parentheses is nonnegative, each is less than ε. Hence $\int_a^b f$ and $\int_b^c f$ exist.

Now in either of these cases, we have

$$\int_a^b s_1 \leq \int_a^b f \leq \int_a^b t_1 \quad \text{and} \quad \int_b^c s_2 \leq \int_b^c f \leq \int_b^c t_2,$$

by definition. Adding, we obtain

$$\int_a^c s = \int_a^b s_1 + \int_b^c s_2 \leq \boxed{\int_a^b f + \int_b^c f} \leq \int_a^b t_1 + \int_b^c t_2 = \int_a^c t.$$

Since the expression in the box lies between $\int_a^c s$ and $\int_a^c t$, the Riemann condition implies that it equals $\int_a^c f$.

<u>Step 5.</u> We prove the comparison property (3).

Consider the set of all step functions s such that $s \leq f$ on $[a,b]$; also consider the set of all step functions t such that $g \leq t$ on $[a,b]$. Because $f \leq g$ on $[a,b]$, we conclude that $s \leq t$ on $[a,b]$, whence

$$\int_a^b s \leq \int_a^b t,$$

<u>because</u> (3) holds for step functions. Holding t fixed and letting s vary, we conclude that

$$\sup \left\{\int_a^b s\right\} \leq \int_a^b t,$$

for any fixed $t \geq g$. <u>Now</u> letting t vary, we see that

$$\sup \left\{\int_a^b s\right\} \leq \inf \left\{\int_a^b t\right\}.$$

That is,

$$\underline{I}(f) \leq \overline{I}(g).$$

Since both f and g are integrable, we have $\underline{I}(f) = \int_a^b f$ and $\overline{I}(g) = \int_a^b g$, so our result is proved.

<u>Step 6.</u> Finally, we prove the reflection property (4).

Given $\varepsilon > 0$, choose step functions s and t so that $s \leqslant f \leqslant t$ on $[a,b]$ and $\int_a^b t - \int_a^b s < \varepsilon$. Then $s(-x)$ and $t(-x)$ are step functions on $[-b,-a]$, and

$$s(-x) \leqslant f(-x) \leqslant t(-x)$$

on $[-b,-a]$. Now

$$\int_{-b}^{-a} t(-x) - \int_{-b}^{-a} s(-x) = \int_a^b t - \int_a^b s < \varepsilon;$$

here we use the fact that (4) holds for step functions. Thus

$$\int_{-b}^{-a} f(-x)$$

exists., by the Riemann condition. Using (4) for step functions again,

$$\int_a^b s = \int_{-b}^{-a} s(-x) \leqslant \boxed{\int_{-b}^{-a} f(-x)} \leqslant \int_{-b}^{-a} t(-x) = \int_a^b t.$$

Since the expression in the box lies between the integrals of s and t, it must by the Riemann condition equal the integral of f.

Exercises

1. Prove property (1) for step functions. [Hint: If s and t are step functions, the first thing to do is to choose a partition P that is compatible with both s and t. Then show $cs + dt$ is a step function compatible with P.]

2. Prove property (2) for step functions. [Hint: If P_1 is a partition of $[a,b]$ and P_2 is a partition of $[b,c]$, then $P_1 \cup P_2$ is a partition of $[a,c]$.]

3. We know (2) holds if $a < b < c$. Show that with our convention, it holds in all cases:

$a = b$,	$a < c < b$,	$c < a < b$,
$a = c$,	$b < a < c$,	$c < b < a$.
$b = c$,	$b < c < a$,	

4. Let $x_0 < \cdots < x_n$ be a partition of $[a,b]$. Let s be a step function on $[a,b]$ such that $s(x) = s_k$ for $x_{k-1} < x < x_k$. Let h be an increasing function on $[a,b]$. Suppose we define

$$\int_a^b s\,dh = \sum_{k=1}^n s_k \cdot (h(x_k) - h(x_{k-1})).$$

(a) Show that this integral is well-defined.

(b) Show that this integral satisfies the linearity, additivity, and comparison properties. You need to use the fact that h is increasing in order to prove one of these properties; which one?

[This definition is actually an important one in mathematics. It leads to a generalization of the integral called the Riemann–Stieltjes integral; one defines $\int_a^b f\,dh$ by using upper and lower integrals, just as before. This integral is important in probability theory.]

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