3.3 Continuity of the square root function.

The following theorem shows that the square-root function is continuous for $x \geq 0$. We will give a different proof, based on the intermediate-value theorem, shortly.

Theorem. (i) $\lim_{x \to 0+} \sqrt{x} = 0.$

(ii) If $a > 0$, $\lim_{x \to a} \sqrt{x} = \sqrt{a}.$

Proof. (i) Given $\varepsilon > 0$, we wish to ensure that $|\sqrt{x}-0| < \varepsilon$. This will occur if $x < \varepsilon^2$. So the choice $\delta = \varepsilon^2$ will work; if $0 < x < \varepsilon^2$, then $\sqrt{x} < \varepsilon$.

(ii) Given $\varepsilon > 0$, we wish to ensure that

$$|\sqrt{x} - \sqrt{a}| < \varepsilon.$$

But

$$|\sqrt{x} - \sqrt{a}| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} \leq \frac{|x-a|}{\sqrt{a}} .$$

So we need merely choose $\delta = \varepsilon\sqrt{a}$; if $|x-a| < \varepsilon\sqrt{a}$, then $|\sqrt{x}-\sqrt{a}| < \varepsilon$. □

Exercises on continuity

1. Show <u>directly</u> <u>from</u> <u>the</u> <u>definition</u> that $f(x) = 1/x$ is continuous at $x = 3$. (That is, given $\epsilon > 0$, define a $\delta > 0$ and show it will work.)

2. Let $f(x)$ be defined for all x, and continuous except for $x = -1$ and $x = 3$. Let

$$g(x) = \begin{cases} x^2 + 1 & \text{for } x > 0, \\ x - 3 & \text{for } x \leq 0. \end{cases}$$

For what values of x can you be sure that $f(g(x))$ is continuous? Explain.