The exponential and logarithm functions

In this section, we study the exponential and logarithm functions and derive their properties.

We also define a^b for $a > 0$ and b arbitrary, and we verify the laws of exponents.

As we did for the trig functions, we shall assume a theorem concerning the existence of the exponential function, postponing the proof until after we have studied power series. Thus we assume the following:

Theorem 1. There exists a function $E(x)$, defined for all real numbers x, satisfying the following conditions

$$E'(x) = E(x); \;\; E(0) = 1.$$

We call E the exponential function, for reasons to be seen. It is sometimes denoted $\exp(x)$.

Theorem 2. (i) The equation

$$E(a+b) = E(a)E(b)$$

holds for all a and b. In particular, $E(a)E(-a) = 1$ for all a.

(ii) $E(x)$ is continuous, positive, and strictly increasing.

(iii) The conditions $E'(x) = E(x)$ and $E(0) = 1$ determine $E(x)$ uniquely.

(iv) If n is an integer and a is a real number, then

$$E(na) = E(a)^n.$$

In particular, if e is defined by the equation $e = E(1)$, then

$$E(n) = e^n.$$

This equation shows why E is called the "exponential function".

(v) The number e satisfies the inequalities

$$2 \leq e \leq 4.$$

(vi) E(x) takes on every positive real value exactly once.

<u>Proof.</u> (i) For fixed b, let us set

$$f(x) = E(x+b)E(-x).$$

Then

$$\begin{aligned} f'(x) &= E'(x+b)E(-x) - E(x+b)E'(-x) \\ &= E(x+b)E(-x) - E(x+b)E(-x) \\ &= 0. \end{aligned}$$

Hence f equals a constant K. Setting $x = 0$, we see that $K = E(b)$. Thus

$$(*) \qquad E(x+b)E(-x) = E(b)$$

for all x and b.

If we set $b = 0$ in equation (*), we obtain the equation

$$E(x)E(-x) = 1,$$

which holds for all x. If then we multiply both sides of equation (*) by E(x), we obtain the equation

$$E(x)E(x+b)E(-x) = E(x)E(b), \quad \text{or}$$

$$E(x+b) = E(x)E(b).$$

Setting $x = a$ gives our desired equation.

(ii) E(x) is continuous because it is differentiable. The equation

$$E(x)E(-x) = 1$$

implies that $E(x) \neq 0$ for all x. The intermediate-value theorem then applies to show that, since E(x) is positive for $x = 0$, it is positive for all x. It follows that $E'(x) = E(x)$ is positive for all x, so that E is strictly increasing.

(iii) Let $\overline{E}(x)$ be another function satisfying the given conditions. Set

$$g(x) = \overline{E}(x)E(-x).$$

One checks readily that $g'(x) = 0$, so that $g(x)$ is constant. Setting $x = 0$, we see this constant is 1. Hence $\overline{E}(x)E(-x) = 1$, or $\overline{E}(x) = E(x)$.

(iv) One proves the result for positive integers by induction: The equation

$$E(na) = E(a)^n$$

holds for $n = 1$, trivially. If it holds for n, compute

$$\begin{aligned} E((n+1)a) &= E(na+a) \\ &= E(na)E(a) \quad \text{by (i)} \\ &= E(a)^n E(a) \quad \text{by the induction hypothesis,} \\ &= E(a)^{n+1}. \end{aligned}$$

The equation holds when $n = 0$ by definition (both sides equal 1), and it holds for negative integers because $E(na)\,E(-na) = 1$, so that

$$E(-na) = 1/E(na) = 1/E(a)^n.$$

(v) Because E is increasing, $E(x) \geq 1$ for $x \geq 0$. The comparison theorem implies that

$$1 \leq \int_0^1 E(x)dx = \int_0^1 E'(x)dx = E(1) - E(0) = e - 1.$$

Hence $e \geq 2$. We leave the other inequality as an exercise.

(vi) It follows from what we have proved that

$$E(n) = e^n \geq 2^n, \text{ and}$$

$$E(-n) = 1/E(n) \leq 1/2^n.$$

Given any positive real number r, we may choose a positive integer n such that

$1/2^n \le r \le 2^n$. The intermediate-value theorem then implies that E(x) takes on the value r for some x in the interval $[-n,n]$. □

Remark. Since $e^n = E(n)$ for all integers n, it seems reasonable to define e^x for arbitrary real x by the equation $e^x = E(x)$. Theorems 1 and 2 can then be restated in this new notation, which is standard, as follows:

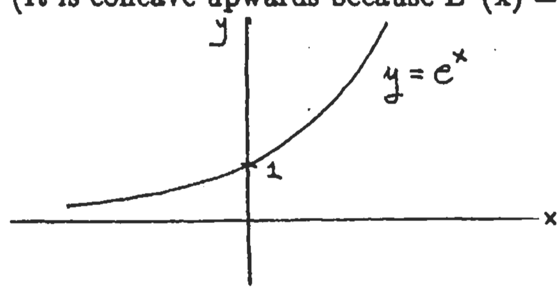
$$\frac{d}{dx}(e^x) = e^x \quad \text{and} \int e^x\,dx = e^x + C.$$

$$e^{a+b} = e^a \cdot e^b,$$

$$e^{na} = (e^a)^n.$$

The latter two equations are special cases of the laws of exponents, which shall prove shortly in full generality.

Remark. The preceding theorem implies that the function $E(x) = e^x$ has the following familiar graph. (It is concave upwards because $E'(x) = E(x) > 0$.)



Exercises 1. Show that

$$\tfrac{1}{2} \le \int_0^{1/2} E(x)dx \le \tfrac{1}{2}\sqrt{e}.$$

Show the integral equals $\sqrt{e} - 1$; conclude that $2.25 \le e \le 4$.

2. Show more generally, by integrating E(x) over the interval $[0,1/n]$, that

$$1/n \le \sqrt[n]{e} - 1 \le \sqrt[n]{e}/n.$$

Conclude that

$$(1+\tfrac{1}{n})^n \le e \le (1+\tfrac{1}{n})^{n+1}.$$

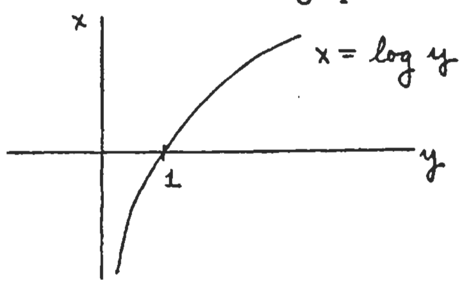
These inequalities give a (not very useful) way of computing e. Try $n = 2$ and ~~$n = 3$~~ $n=4$ in these formulas, using your calculator.

The logarithm function.

Definition. The function e^x is strictly increasing and takes on each positive value exactly once. We define the (natural) logarithm function to be its inverse. That is, if y is any positive number, we define

$$\log y = x \quad \text{if and only if} \quad y = e^x.$$

The logarithm function thus has the graph



It is strictly increasing and continuous. It is defined only for $y > 0$, and it takes on every real value.

The fact that these functions are inverses of each other implies that:

$$e^{\log y} = y \quad \text{if } y > 0,$$
$$\log(e^x) = x \quad \text{for all } x.$$

Theorem 3. The logarithm function has the following properties:

(i) $\frac{d}{dx}(\log x) = \frac{1}{x}$ and

$$\int \frac{1}{x}\,dx = \log |x| + C.$$

(ii) $\log (ab) = \log a + \log b$ if a and b are positive.

(iii) $\log(a^n) = n \log a$ if n is an integer and a is positive.

Proof (i) Let $f(x) = e^x$ and $g(y) = \log y$. Then g is the inverse function to f. We use the formula for the derivative of an inverse function:

$$g'(y) = \frac{1}{f'(g(y))}.$$

Now $f'(x) = f(x)$ for all x, so

$$g'(y) = \frac{1}{f'(g(y))} = \frac{1}{e^{\log y}} = \frac{1}{y}.$$

since $f(g(y)) = e^{\log y} = y$.

If x is positive, the derivative of $\log|x| = \log x$ is $1/x$. If x is negative, the derivative of $\log|x| = \log(-x)$ is $(-1)/(-x) = 1/x$. Thus $\int dx/x = \log|x| + C$.

(ii) Given a and b, let

$$x = \log a \quad \text{and} \quad y = \log b.$$

We have the equation

$$e^{x+y} = e^x \cdot e^y.$$

Since both sides of this equation are positive, we can take their logs to conclude that

$$x + y = \log(e^x \cdot e^y),$$

so that

$$\log a + \log b = \log(a \cdot b),$$

as desired.

(iv) Given $a > 0$, let $x = \log a$. Then $a = e^x$, so

$$a^n = (e^x)^n = e^{nx} \quad \text{by (iv) of Theorem 2,}$$

so that

$$\log a^n = nx = n(\log a). \quad \square$$

Exponents

Theorem 4. There is one and only one function a^x, defined for all positive a and all real x, such that the following four conditions hold:

(i) The function a^x is positive and continuous.

(ii) $a^1 = a$.

(iii) $a^{x+y} = a^x a^y$.

(iv) $(a^x)^y = a^{xy}$.

This function satisfies the following additional conditions:

(v) $a^x b^x = (ab)^x$.

(vi) $\log a^x = x \log a$.

Proof. Uniqueness. Suppose a^x is defined and satisfies conditions (i) – (iv). Conditions (ii) and (iii) imply that

$$a^1 = a \quad \text{and} \quad a^{n+1} = a^n \cdot a$$

for every positive integer n. The equation

$$a^1 \cdot a^0 = a^{1+0} = a^1$$

implies (since $a > 0$) that $a^0 = 1$. Finally, the equation

$$a^n a^{-n} = a^0 = 1$$

implies that

$$a^{-n} = 1/a^n.$$

Hence integral powers of a must be defined as we have defined them earlier.

Now if n is a positive integer and m is any integer, (iv) implies that

$$(a^{1/n})^n = a \quad \text{or} \quad a^{1/n} = \sqrt[n]{a} .$$

Then, using (iv) again, we see that

$$(a^{m/n}) = (a^{1/n})^m = (\sqrt[n]{a})^m .$$

Thus for x rational, a^x is completely determined by conditions (ii) – (iv) and positivity.

Continuity now implies that a^x is determined for all x: Suppose f(x) and g(x) are two functions that satisfy (i)–(iv). Let x_0 be arbitrary. Given $\epsilon > 0$, choose δ so that

$|f(x)-f(x_0)| < \epsilon$ and $|g(x)-g(x_0)| < \epsilon$ for $|x-x_0| < \delta$. Then choose x_1 rational with $|x_1-x_0| < \delta$. It follows from what we have already showed that $f(x_1) = g(x_1)$. Then $|f(x_0)-g(x_0)| < 2\epsilon$. Since ϵ is arbitrary, we must have $f(x_0) = g(x_0)$.

Existence. We motivate the definition as follows: If n is a positive integer, then

$$\log a = \log(\sqrt[n]{a})^n = n \log \sqrt[n]{a},$$

so that

$$\log (\sqrt[n]{a})^m = m \log \sqrt[n]{a} = \frac{m}{n} \log a,$$

or

$$(\sqrt[n]{a})^m = E(\frac{m}{n} \log a).$$

This equation suggests the following definition.

We define, for arbitrary x,

$$a^x = E(x \log a).$$

Then condition (vi) holds trivially, for $\log a^x = x \log a$ by definition.

We show that the other conditions of the theorem are satisfied:

(i) a^x is positive and continuous.

(ii) $a^1 = E(\log a) = a$.

(iii)
$$\begin{aligned} a^{x+y} &= E((x+y) \log a) && \text{by definition,} \\ &= E(x \log a + y \log a) && \text{by distributivity,} \\ &= E(x \log a) \cdot E(y \log a) && \text{by Theorem 2,} \\ &= a^x \cdot a^y. && \text{by definition.} \end{aligned}$$

(iv) $(a^x)^y = E(y \log a^x)$ by definition,

$= E(y(x \log a))$ by (vi),

$= E((xy) \log a)$ by associativity,

$= a^{xy}$. by definition.

(v) $(ab)^x = E(x \log (ab))$ by definition,

$= E(x(\log a + \log b))$ by Theorem 3,

$= E(x \log a + x \log b)$ by distributivity,

$= E(x \log a) \cdot E(x \log b)$ by Theorem 2,

$= a^x \cdot b^x$. by definition. □

Theorem 5. Let c be a real constant. Then

$$D(x^c) = cx^{c-1} \quad \text{if } x > 0,$$

$$\int x^c \, dx = \frac{x^{c+1}}{c+1} + C \quad \text{if } c \neq -1 \text{ and } x > 0.$$

Proof. Since $x^c = E(c \log x)$, we can use the chain rule. We have

$$\begin{aligned} D(x^c) &= E(c \log x)D(c \log x) \\ &= E(c \log x)c/x \\ &= x^c(c/x) = cx^{c-1}. \end{aligned}$$

The integration formula follows at once. □

Theorem 6. Let a be a real constant. a > 0. Then

$$D(a^x) = a^x \log a,$$

$$\int a^x \, dx = \frac{a^x}{\log a} + C \quad \text{if } a \neq 1.$$

The proof is left as an exercise

For other differentiation and integration formulas involving logarithms and exponentials, see 6.7 and 6.16 of Apostol.

Remark concerning common logarithms.

The logarithm function we have defined is sometimes called the "natural logarithm". A different version of the logarithm was once useful. It can be obtained as follows: Consider the function

$$f(x) = 10^x = E(x \log 10).$$

It is strictly increasing, since by the chain rule,

$$f'(x) = E(x \log 10) \cdot \log 10,$$

which is positive. Furthermore, since x log 10 takes on all real values, E(x log 10) = f(x) takes on all positive values. The inverse of f is called the "common logarithm" or the "logarithm to the base 10", and denoted by $\log_{10} y$. That is, if $y > 0$, we define

$$\log_{10} y = x \text{ if and only if } y = 10^x.$$

This function was at one time useful for computational purposes, but it has long since fallen into oblivion.

A similar remark applies to obtain logarithms to other bases. If b is any positive number with $b \neq 1$, one defines

$$\log_b y = x \quad \text{if and only if } y = b^x.$$

Remark on motivation

Just as the sine and cosine functions arise most naturally as the fundamental solutions of the differential equation for simple harmonic motion, so the exponential function arises most naturally from consideration of the important differential equation

$$f'(x) = kf(x),$$

called the "equation of population growth (or decay)." If k is, for instance, the difference of the birth and death rate (per thousand, say, of a population) in a given time period, then this is the equation for the actual population (in thousands), as a function of time.

One checks at once that the function e^{kx} satisfies this equation. More generally, every solution of this equation can be expressed in terms of e^{kx}:

Theorem 7. Suppose f(x) is defined for all x and satisfies the equation

$$f'(x) = kf(x).$$

Let f(0) = a. Then

$$f(x) = ae^{kx}.$$

Proof. Let us set

$$g(x) = f(x)E(-kx).$$

Then we compute

$$g'(x) = f'(x)E(-kx) - kf(x)E'(-kx) = 0,$$

so g is constant. Since $g(0) = f(0) \cdot E(0) = a$,

$$g(x) = a.$$

Multiplying by E(kx), we have f(x) = aE(kx), as desired. □

Exercises

1. If a is constant, show that in general

$$D(a^x) \neq xa^{x-1}, \text{ and}$$

$$\int a^x \, dx \neq \frac{a^{x+1}}{x+1}.$$

[Anyone who makes the mistake, on a quiz, of thinking these are equalities gets clobbered!]

2. Evaluate $\int_0^1 \pi^x \, dx$ and $\int_0^1 x^\pi \, dx$.