Example 1. $f(x) = \dfrac{1}{x}$

We'll find the derivative of the function $f(x) = \frac{1}{x}$. To do this we will use the formula:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

Graphically, we will be finding the slope of the tangent line at at an arbitrary point $(x_0, \frac{1}{x_0})$ on the graph of $y = \frac{1}{x}$. (The graph of $y = \frac{1}{x}$ is a hyperbola in the same way that the graph of $y = x^2$ is a parabola.)

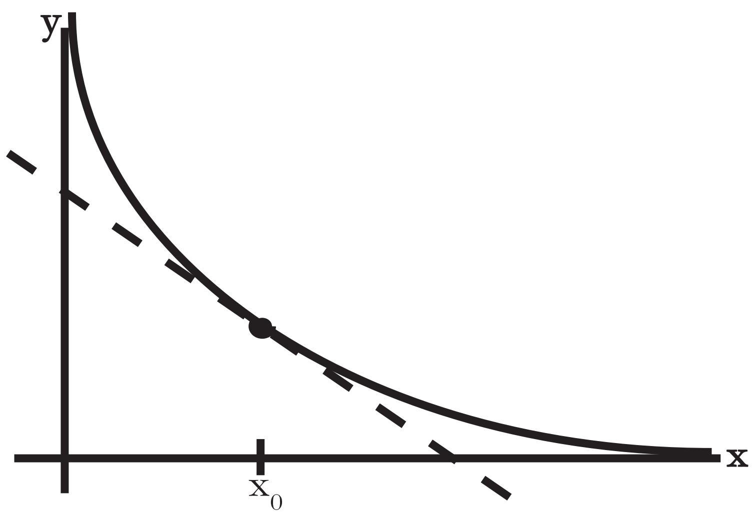

Figure 1: Graph of $\frac{1}{x}$

We start by computing the slope of the secant line:

$$\begin{aligned}
\frac{\Delta f}{\Delta x} &= \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \\
&= \frac{\frac{1}{x_0 + \Delta x} - \frac{1}{x_0}}{\Delta x} \\
&= \frac{(x_0)(x_0 + \Delta x)}{(x_0)(x_0 + \Delta x)} \frac{\frac{1}{x_0 + \Delta x} - \frac{1}{x_0}}{\Delta x} \\
&= \frac{\frac{(x_0)(x_0 + \Delta x)}{x_0 + \Delta x} - \frac{(x_0)(x_0 + \Delta x)}{x_0}}{(x_0)(x_0 + \Delta x)\Delta x} \\
&= \frac{1}{\Delta x} \frac{x_0 - (x_0 + \Delta x)}{(x_0)(x_0 + \Delta x)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Delta x}\frac{-\Delta x}{(x_0)(x_0+\Delta x)} \\
&= \frac{-1}{(x_0)(x_0+\Delta x)}.
\end{aligned}
$$

Next, we see what happens to the slopes of the secant lines as Δx tends to zero:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{-1}{(x_0)(x_0+\Delta x)} = \frac{-1}{x_0^2}$$

One thing to keep in mind when working with derivatives: it may be tempting to plug in $\Delta x = 0$ right away. If you do this, however, you will always end up with $\dfrac{\Delta f}{\Delta x} = \dfrac{0}{0}$. You will always need to do some cancellation to get at the answer.

We've computed that $f'(x) = \frac{-1}{x_0^2}$. Is this correct? How might we check our work? First of all, $f'(x_0)$ is negative — as is the slope of the tangent line on the graph of $y = \frac{1}{x}$. Secondly, as $x_0 \to \infty$ (i.e. as x_0 grows larger and larger), the tangent line is less and less steep. So $\frac{1}{x_0^2}$ should get closer to 0 as x_0 increases, which it does.

Question: Explain why $\displaystyle\lim_{\Delta x \to 0} \frac{-1}{(x_0)(x_0+\Delta x)} = \frac{-1}{x_0^2}$ again?

Answer: The point x_0 could be any point; let's suppose that $x_0 = 3$ so that we can look at this limit in a specific case.

We want to know the value of $\frac{-1}{(3)(3+\Delta x)}$ as Δx tends toward zero. As Δx gets smaller and smaller $3 + \Delta x$ gets closer and closer to 3, and so $\frac{-1}{(3)(3+\Delta x)}$ gets closer and closer to $\frac{-1}{(3)(3)} = \frac{-1}{9}$.

Question: Why is it that $\dfrac{\frac{1}{x_0+\Delta x} - \frac{1}{x_0}}{\Delta x} = \dfrac{1}{\Delta x}\dfrac{x_0 - (x_0+\Delta x)}{(x_0)(x_0+\Delta x)}$?

Answer: There are two steps in this simplification. We factored out the Δx that was in the denominator to become the $\frac{1}{\Delta x}$ "out front". At the same time, we rewrote the difference of two fractions $\frac{1}{x_0+\Delta x} - \frac{1}{x_0}$ using a common denominator.

This common denominator was $(x_0)(x_0 + \Delta x)$, which is just the product of the denominators in $\frac{1}{x_0+\Delta x} - \frac{1}{x_0}$. To get the common denominator, we multiply the first fraction by $\frac{(x_0)}{(x_0)} = 1$ and the second by $\frac{(x_0+\Delta x)}{(x_0+\Delta x)}$. (Multiplying by 1 won't change its value, but can change the algebraic expression we use to describe that value.) The denominators cancel, as intended, and we're left with $\frac{x_0-(x_0+\Delta x)}{(x_0)(x_0+\Delta x)}$.