Quadratic Approximation at 0 for Several Examples

We'll save the derivation of the formula:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 \quad (x \approx x_0)$$

for later; right now we're going to find formulas for quadratic approximations of the functions for which we have a library of linear approximations.

Basic Quadratic Approximations:

In order to find quadratic approximations we need to compute second derivatives of the functions we're interested in:

$f(x)$	$f'(x)$	$f''(x)$	$f(0)$	$f'(0)$	$f''(0)$
$\sin x$	$\cos x$	$-\sin x$	0	1	0
$\cos x$	$-\sin x$	$-\cos x$	1	0	-1
e^x	e^x	3^x	1	1	1
$\ln(1+x)$	$\frac{1}{1+x}$	$\frac{-1}{(1+x)^2}$	0	1	-1
$(1+x)^r$	$r(1+x)^{r-1}$	$r(r-1)(1+x)^{r-2}$	1	r	$r(r-1)$.

Plugging the values for $f(0)$, $f'(0)$ and $f''(0)$ in to the quadratic approximation we get:

1. $\sin x \approx x \quad (\text{if } x \approx 0)$
2. $\cos x \approx 1 - \frac{x^2}{2} \quad (\text{if } x \approx 0)$
3. $e^x \approx 1 + x + \frac{1}{2}x^2 \quad (\text{if } x \approx 0)$
4. $\ln(1+x) \approx x - \frac{1}{2}x^2 \quad (\text{if } x \approx 0)$
5. $(1+x)^r \approx 1 + rx + \frac{r(r-1)}{2}x^2 \quad (\text{if } x \approx 0)$

We've computed some formulas; now let's think about their meaning.

Geometric significance (of the quadratic term)

A quadratic approximation gives a best-fit parabola to a function. For example, let's consider $f(x) = \cos(x)$ (see Figure 1).

The linear approximation of $\cos x$ near $x_0 = 0$ approximates the graph of the cosine function by the straight horizontal line $y = 1$. This doesn't seem like a very good approximation.

The quadratic approximation to the graph of $\cos(x)$ is a parabola that opens downward; this is much closer to the shape of the graph at $x_0 = 0$ than the line

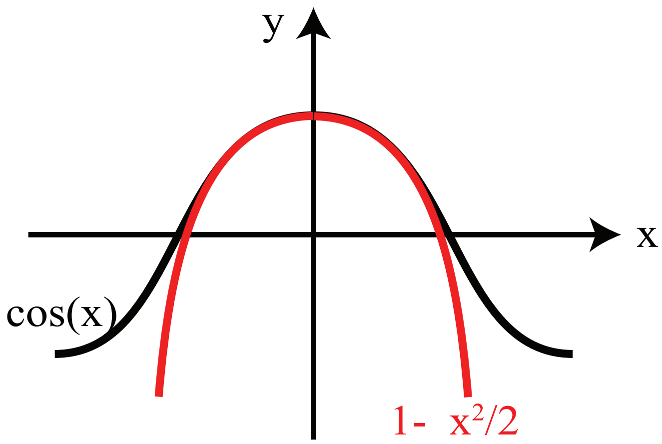

Figure 1: Quadratic approximation to $\cos(x)$.

$y = 1$. To find the equation of this quadratic approximation we set $x_0 = 0$ and perform the following calculations:

$$\begin{aligned} f(x) = \cos(x) \quad &\implies \quad f(0) = \cos(0) = 1 \\ f'(x) = -\sin(x) \quad &\implies \quad f'(0) = -\sin(0) = 0 \\ f''(x) = -\cos(x) \quad &\implies \quad f''(0) = -\cos(0) = -1. \end{aligned}$$

We conclude that:

$$\cos(x) \approx 1 + 0 \cdot x - \frac{1}{2}x^2 = 1 - \frac{1}{2}x^2.$$

This is the closest (or "best fit") parabola to the graph of $\cos(x)$ when x is near 0.