Curve Sketching Example 1

Example 1: Sketch the graph of $f(x) = 3x - x^3$.

First we note that:

$$f'(x) = 3 - 3x^2 = 3(1 - x)(1 + x)$$

We can see that if $-1 < x < 1$, both $(1 - x)$ and $(1 + x)$ are positive, so $f'(x)$ must also be positive, so f is increasing (by our first principle.)

When $x > 1$, $f'(x) < 0$ and f is decreasing. When $x < -1$, then $(1 - x)$ is positive and $(1 + x)$ is negative, so $f'(x) = 3(1 - x)(1 + x)$ is negative and f is decreasing.
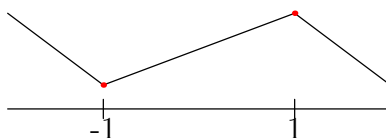

Figure 1: Turning points of $f(x) = 3x - x^3$.

We get a rough schematic of the graph of the function by drawing a number line at the bottom of our page as shown in Figure 1. Above the interval from negative infinity to -1, draw a diagonal line slanting down and to the right — this is one of the intervals on which f is decreasing. Above the interval from -1 to 1 draw a line slanting up and to the right — f is increasing. Finally, draw a line slanting down and to the right above the interval from 1 to positive infinity. The resulting zigzag gives us an idea of what an accurately drawn graph might look like.

We immediately notice two important features of the function lie at the points where the graph changes direction. These places where the derivative changes sign are turning points in the graph.

Definition: If $f'(x_0) = 0$, we call x_0 a *critical point* and $y_0 = f(x_0)$ is a *critical value* of f.

Our next step in understanding the graph of $f(x)$ is to plot the critical points and values of f. The critical points are $x = 1$ and $x = -1$; these are the values of x for which $(x - 1)$ and $(x + 1)$ are zero. The critical values are $f(1) = 3 \cdot 1 - 1^3 = 2$ and $f(-1) = 3 \cdot (-1) - (-1)^3 = -2$.

We plot the two points $(-1, -2)$ and $(1, 2)$, which we know are on the graph of f. Because we know where f' is positive and where it is negative, we also know that the graph decreases to $(-1, -2)$ and then starts to increase and that it increases toward $(1, 2)$ and then starts to decrease. In other words, the graph is shaped like a smile near $(-1, -2)$ and like a frown near $(1, 2)$. (We could also learn this from the second derivative.)

If we notice that $f(0) = 0$, we can now guess what the entire graph might look like but we still don't know for sure how far it decreases or increases to the left and right.

We can also notice that because all the powers of x are odd, the function f is odd; $f(-x) = 3(-x) - (-x)^3 = -3x + x^3 = -f(x)$. This means that if we can graph the function accurately for $x > 0$ we can reflect the graph across the y-axis to get a graph of the entire function.

In general, you should use precalculus skills to get information like $f(0) = 0$ or "f is odd" as much as you can.

The final detail we need to worry about is what happens at the "ends" of the graph. This topic is often neglected, but if you're using a graphing calculator or computer this can be the hardest part of understanding the graph of a function.

We want to know what happens as $x \to \infty$ and as $x \to -\infty$. As $x \to \infty$, the value of $-x^3$ grows very rapidly while the value of $3x$ is much smaller. So:

$$f(x) \approx -x^3 \to -\infty \text{ as } x \to \infty.$$

Similarly,

$$f(x) \approx x^3 \to \infty \text{ as } x \to -\infty.$$

We now know how far $f(x)$ increases or decreases as $x \to \pm\infty$ — it goes off toward infinity rather than, say, hugging the x-axis.

We now know a lot about the shape of the graph, but we can learn a little bit more about it by looking at the second derivative $f''(x) = -6x$. From this we learn that $f''(x) < 0$ when $x > 0$ and $f''(x) > 0$ when $x < 0$, so the graph is concave down to the right of the y-axis and concave up on the left. The value $x = 0$ is of interest not only because $f(0) = 0$ but also because it is an *inflection point* — a value x_0 for which $f''(x_0) = 0$.

We can now combine everything we've learned to get something like the graph shown in Figure 2.
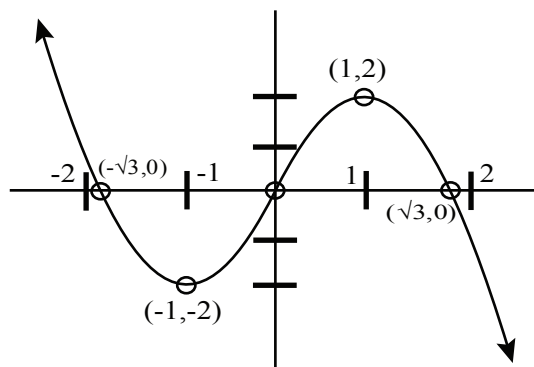

Figure 2: Sketch of the function $y = 3x - x^3$.

Question: What if the graph had a sharp point like the one in the schematic?
Answer: Points like that aren't called critical points, but they are very important. We'll talk about them later.

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