## Cube Root of $x$

Show that for any non-zero starting point $x_{0}$, Newton's method will never find the exact value $x$ for which $x^{3}=0$.

## Solution

Given a value $x_{0}$ close to some $x$ where $f(x)=0$, Newton's method usually produces a series of values $x_{0}, x_{1}, x_{2}, \ldots$ whose values approach $x$. However, there is no guarantee that any of those values $x_{i}$ actually equal $x$.

Newton's method generates a sequence according to the formula:

$$
x_{k+1}=x_{k}-\frac{f\left(x_{k}\right)}{f^{\prime}\left(x_{k}\right)}
$$

In our example, $f(x)=x^{3}$ and $f^{\prime}(x)=3 x^{2}$, so:

$$
\begin{aligned}
x_{k+1} & =x_{k}-\frac{x_{k}^{3}}{3 x_{k}^{2}} \\
& =x_{k}-\frac{x_{k}}{3} \\
x_{k+1} & =\frac{2}{3} x_{k}
\end{aligned}
$$

Starting with $x_{0}$, Newton's method generates the sequence:

$$
\begin{aligned}
x_{1} & =\frac{2}{3} x_{0} \\
x_{2} & =\frac{2}{3} x_{1}=\frac{2}{3} \cdot\left(\frac{2}{3} x_{0}\right)=\frac{2^{2}}{3^{2}} x_{0} \\
x_{3} & =\frac{2}{3} x_{2}=\frac{2^{3}}{3^{3}} x_{0} \\
x_{4} & =\frac{2}{3} x_{3}=\frac{2^{4}}{3^{4}} x_{0} \\
& \vdots \\
x_{n} & =\frac{2^{n}}{3^{n}} x_{0}
\end{aligned}
$$

If $x_{0} \neq 0$, the values of the $x_{i}$ get closer and closer to the desired value 0 but never exactly equal zero.

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### 18.01SC Single Variable Calculus] []

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