

**CHRISTINE
BREINER:**

Welcome back to recitation. In this video I want to explain to you where the coefficients we saw in Simpson's rule actually come from. So what I'm going to do is take this curve that I have and show you what the picture of Simpson's rule does. And then I'm actually going to determine explicitly where the coefficients come from.

So let's look at this picture first. The picture I have here, the white curve is going to be my y equals f of x function. And so, if you remember, what Simpson's rule is saying, you have to have two delta x 's. So in this case h is equal to delta x . So what Simpson's rule is saying is I can find, approximate, the area under this curve from minus h to h by a certain expression. And the expression is, I said delta x is equal to h , so let me write h over 3 times y_0 plus $4y_1$ plus y_2 -- oh, y_2 . So that's what you got--

that's what you saw in the lecture, that this is what the coefficients are. So I want to explain why this is a 1, why that's a 4, why that's a 1, and where that 3 comes from. The picture is not going to explain it, but the picture will help us understand how to go.

So what Simpson's rule does is it takes those three points, so the (x_0, y_0) the (x_1, y_1) and the (x_2, y_2) that are on the curve y equals f of x . And it finds a parabola through those three points. So I'm going to do my best to draw what looks like a parabola through those three points. Something-- I'll draw it lightly first and then I'll fill it in. Something along these lines. Something like that. That's a, sort of looks like a parabola.

And so the idea is Simpson's rule is saying I can find the area under the blue curve. So I can find the area under the blue curve. So this is a function, we'll call this y equal P of x . And that's what you were actually told about in lecture. That you're approximating your function y equals f of x by a quadratic function that goes through the values (x_0, y_0) , (x_1, y_1) , and (x_2, y_2) . And then you find the area under that parabola. Between minus h and h .

Now this picture, you might say, Christine, this looks really like a bad estimate. This looks kind of stupid. Maybe you should have picked a different function to estimate this. And I did this one because I wanted to be zoomed out far and to show you, to give you a little intuition about what happens when we make h smaller. Notice that if I cut the size of h in half, so if I started with here was minus h to h , what would my three points be? My three points would be this point, y_1 , and y_2 would be right here. Well the quadratic through those points is much closer to the actual function.

And if I cut that in half again, I'd have points here, here, and here. Something like that, and that starts to look almost exactly like a quadratic. So if I found the area under this-- or if I wanted to estimate the area under this piece of the curve using a quadratic through those three points, they would be very close. So don't be alarmed by Simpson's rule as an approximation based on this large picture.

So now, what do we have to do? Remember, what's our goal? Our goal is to figure out where the coefficients come from. So what I actually need to do is I need to evaluate a certain integral. Or I need to integrate a certain function. I need to integrate P of x . So what I'm going to be finding through the rest of this video, is the integral from minus h to h of P of x dx . And my goal is to show that this integral is equal to this expression. I want to show that these are equal. That's my goal.

So let's get down to it. What do I know? What's the only thing I know right away about P of x ? I know I'm choosing it to be a quadratic function and I know it goes through three certain points. Right? I'll write down what the three points are when we need them. But first let's just say, in general, let's take this integral for a general quadratic and see how much information we actually need.

So let me come over here. Actually, let me say first, what do I mean my general quadratic? I mean something of the form big A , capital A , x squared plus Bx plus C . So I'm not filling in values for these coefficients yet. Those coefficients are coming exactly from my three points. (x_0, y_0) , (x_1, y_1) , and (x_2, y_2) . So let's just integrate that from minus h to h first and see what kind of information we need.

So if I come over here. So what do I get when I actually integrate this? Well, I get Ax^3 over 3, plus Bx^2 over 2, plus Cx , and then I have to evaluate from minus h to h . Well, if you were thinking about this, it shouldn't be surprising that when I do this, there's not going to be a B term. Why is that? Well, these two functions are even. Ax^2 and C are even functions. And I'm integrating over something that's symmetric about the y -axis. I'm going from minus h to h . So if you think about Ax^2 , and I'm going from minus h to h , I'm finding the area under a quadratic, from minus h to h , it's going to be twice the area from 0 to h . But Bx , that's a line with slope B . If I wanted to integrate Bx from minus h to h , that's the area, the signed area under the curve, of a line Bx , from minus h to h .

If I just draw a quick picture, what does that look like? It's symmetric with respect to a rotation

there. I'd have this signed area. This is the curve y equals $B \cdot x$. From minus h to h , I get this area is negative and this area is positive and they're equal. So, what I'm about to do, you shouldn't be surprised there won't be a B term. So what do we actually get when we evaluate this? We get $\frac{2A}{3} h^3$, plus $2 \cdot C \cdot h$. That's what we get. You can actually work it all, put in all the h 's and see that, but I knew I was going to have double what was here when I evaluated at h , and nothing from the B term.

So we're getting closer. We're getting closer. We might not look like we're getting closer, but we're getting closer. So let's simplify this expression a little bit. Actually, what I ultimately need to do is I need to figure out C and I need to figure out something over here. I need to actually figure out $A h^2$. And let me explain why I need to figure out $A h^2$. In the end, if I come over back to what I want to show-- let me even box it, so we see it-- I want to show that this integral, which I've started to calculate, is equal to $\frac{h}{3}$ times this quantity. So I want to keep one around, but I want the other, any other powers of h to not be there when I'm done. Right?

I need one power of h there. In fact I need an $\frac{h}{3}$ there. So I think what I'll do is I'll start off and I'll pull out an $\frac{h}{3}$, and then I'll try and figure out if I can get the rest of my expression to look like what's in the parentheses. That's really, ultimately, what I'd like to do.

So let's come back over here. I'm going to pull out an $\frac{h}{3}$ from both of these and we're going to see what we end up with. OK so if I pull out an $\frac{h}{3}$ here, what do I end up with? This is easy. That's $2A h^2$. And this one's a little bit trickier. But I have to multiply by $\frac{3}{3}$ to get a 3 there. So I end up with $6C$. OK. Let's make sure I didn't make any mistakes. If I multiply through here I get $\frac{2A}{3} h^3$. If I multiply through here I get $2h \cdot C$. Looking good so far.

Now I have to figure out A and C , or $A h^2$ and C . Well, C is pretty easy to find. Let's think about why that is. I have this polynomial. The polynomial was equal to-- if we come over here and we look again, it's $A x^2$ plus $B \cdot x$ plus C . And the polynomial has to go through those three points I had. So when x is 0 , P of 0 is C . So let's come back and look at our picture. When x is 0 , what's the output on the white curve? It's y_1 . So C is exactly y_1 .

How am I going to find $A h^2$? Well, what I need to do is use these other two points. I'm going to evaluate the function P of x at minus h . And its output has to be y_0 . And I'm going to evaluate the function P of x at h and its output has to be y_2 . So we're going to come over to

the other side, but that's really what we're doing next. So let's come over here and let's evaluate P of h and P of $-h$. So P of h is $A h^2 + B h + C$. And P of $-h$ is $A h^2 - B h + C$. OK. What else? Again we said this one has to be y_2 , the output, and this one has to be y_0 . Because the output at h has to agree with the output of the function f of x . And its output at h was y_2 . The output at $-h$ of P has to be the same as the output at $-h$ of f . And that was y_0 .

Now this might not look fun yet but what if I add these two things together. What happens? I get $2A h^2$. These drop out. And then I get $2C$ is equal to $y_0 + y_2$. I'm getting closer. I'm getting much closer. Because now notice what I have. I have $2A h^2$, I can isolate that. And that's something that I want to figure out. I want to figure out $2A h^2$. So let's figure out what that is.

$2A h^2$ is equal to $y_0 + y_2$ minus, well what did we say C was? C was the output at x equals 0 . Which is y_1 . So it's $2C$, which is $2y_1$. So now we're very close, we're very close to getting what we want. And that's good, because I'm almost out of room.

So let's take that expression, we were working on this expression right here, and let's start to fill in what we get. We get h^3 times $2A h^2$, which is $y_0 + y_2 - 2y_1$, and then I have to add $6C$. Now what's C ? C is y_1 . So I'm going to add $6y_1$. And if I simplify that, I get y_0 -- bingo-- plus $4y_1$ plus y_2 . And that's what we wanted.

So let me take us through kind of where all this came from again, what the big pieces were and we'll see now we understand how we do Simpson's rule, and these small chunks of Simpson's rule. So let's come back to the very beginning. OK, in the very beginning, what we had was a function f of x , that was the white curve. y equals f of x was the white curve. And then I found a quadratic that went through $-h, y_0, (0, y_1)$, and (h, y_2) . Went through those three points. And I called that quadratic P of x .

And then what I was doing, we know Simpson's rule says find the area under the curve of P of x between $-h$ and h . So what I did was I came over here and I said OK, P of x , integral of P of x from $-h$ to h . I wrote P of x in a very general form. And then I actually found an integral. I came through and after writing it out, calculating the integral, I got to this expression. This is actually the integral of P of x from $-h$ to h . So it's the area under the curve of P of x from $-h$ to h is that.

And now I had to figure this out, how to write this in terms of the outputs of f of x . Which were

y_0 , y_1 , and y_2 . Those were the ones we were interested in. So I ultimately knew I wanted an h over three in front. I knew I wanted my integral in my quadratic to be h over 3 times something, so I pulled out an h over 3, and then I looked at what this expression was. And if I could get this expression to look like the inside of what I wanted, which was y_0 plus $4y_1$ plus y_2 , I was in business. And so then I used outputs that I knew for P of x to find $2A h$ squared and to find C . I know P of h is the output of the f of x function at the far right, which was y_2 . And I knew P of minus h was the output of the f of x function at the far left. That's y_0 . I actually then evaluate P at h and minus h , and when I add those together, I get my $2A h$ squared in terms of y_0 , y_1 , and y_2 . Let me do this in terms of y_0 , y_1 , and y_2 . Because I also knew C was y_1 . We talked about that as well. C had to be y_1 . So I can then do the substitution I need right here and show where the coefficients in Simpson's rule come from. So hopefully that was informative for you. And I think I'll stop there.