Simpson's Rule Approximation of $\int_1^2 \frac{dx}{x}$

If we use Simpson's rule with two intervals to estimate the value of $\int_1^2 \frac{dx}{x}$, the result is surprisingly accurate:

$$\frac{\Delta x}{3}\left(y_0 + 4y_1 + y_2\right) = \frac{1}{6}\left(1 + 4\frac{2}{3} + \frac{1}{2}\right) \approx 0.69444...$$

This is impressively close to our calculator's estimate of 0.693147. (Our calculator uses more than two intervals!)

The accuracy of Simpson's rule is proportional to Δx^4. In this example, that means that if we'd used 10 intervals the error would be about 10^{-4}; we'd have four digit accuracy. The calculation isn't even particularly difficult — it's possible to do it by hand. The other rules for numerical approximation aren't as accurate.

The reason Simpson's rule is more accurate is that it's matching a parabola to the curve, rather than a straight line. Simpson's rule gives the exact area beneath the graphs of functions of degree two or less (parabolas and straight lines), while the other methods are only exact for functions whose graphs are linear.

Oddly, Simpson's rule also gives the exact area under the curve for cubic curves, which explains why the error is comparable to the fourth power of Δx.

However, Simpson's rule does have problems if the derivative is unbounded (e.g. near 0 for the function $f(x) = \frac{1}{x}$) or if the graph is not "smooth" enough.

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