1. Compute the following integral:

$$\int_1^4 \sqrt{t}\, \ln t \, dt$$

We apply integration by parts:

$$\begin{array}{ll} u = \ln t & v = \frac{2}{3}t^{3/2} \\ du = \frac{1}{t}\, dt & dv = \sqrt{t}\, dt \end{array}$$

Then:

$$\begin{aligned} \int_1^4 \sqrt{t}\ln t\, dt &= \frac{2}{3}t^{3/2}\ln t\Big|_1^4 - \int_1^4 \frac{2}{3}\underbrace{t^{3/2}\cdot t^{-1}}_{t^{1/2}}\, dt \\ &= \left[\frac{2}{3}t^{3/2}\ln t - \frac{4}{9}t^{3/2}\right]_1^4 \\ &= \frac{16}{3}\ln 4 - \frac{4}{9}(4^{3/2}-1) \\ &= \frac{16}{3}\ln 4 - \frac{28}{9} \end{aligned}$$

2. Compute the following integral:

$$\int_0^{\pi/4} \tan^4\theta \sec^6\theta\, d\theta$$

Recall that $\sec^2\theta = 1 + \tan^2\theta$, so:

$$\int_0^{\pi/4} \tan^4\theta\sec^6\theta\, d\theta = \int_0^{\pi/4} \tan^4\theta(1+\tan^2\theta)^2\sec^2\theta\, d\theta.$$

Let $u = \tan\theta$. Then $du = \sec^2\theta\, d\theta$ and:

$$0 \le \theta \le \pi/4 \Rightarrow 0 \le u \le 1.$$

$$\begin{aligned} \int_0^{\pi/4} \tan^4\theta\sec^6\theta\, d\theta &= \int_0^1 u^4(1+u^2)^2\, du \\ &= \int_0^1 u^4(1+2u^2+u^4)\, du \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 (u^4 + 2u^6 + u^8)\, du \\
&= \frac{1}{5}u^5 + \frac{2}{7}u^7 + \frac{1}{9}u^9 \Big|_0^1 \\
&= \frac{1}{5} + \frac{2}{7} + \frac{1}{9} = \frac{188}{315}.
\end{aligned}
$$

3. Compute the following integral:

$$\int \frac{10}{(x-1)(x^2+9)}\, dx$$

$$
\begin{aligned}
\int \frac{10}{(x-1)(x^2+9)}\, dx &= \int \frac{1}{x-1} - \frac{x+1}{x^2+9}\, dx \\
&= \int \frac{1}{x-1}\, dx - \int \frac{x}{x^2+9}\, dx - \int \frac{1}{x^2+9}\, dx \\
&= \ln(x-1) - \frac{1}{2}\ln(x^2+9) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c
\end{aligned}
$$

4. Compute the following integral:

$$\int \frac{1}{(5-4x-x^2)^{5/2}}\, dx$$

$$\int \frac{1}{(5-4x-x^2)^{5/2}}\, dx = \int \frac{1}{(9-(2+x)^2)^{5/2}}\, dx$$

Let $x + 2 = 3\sin\theta$. Then $dx = 3\cos\theta\, d\theta$ and:

$$
\begin{aligned}
\int \frac{1}{(9-(2+x)^2)^{5/2}}\, dx &= \int \frac{3\cos\theta}{9^{5/2}(1-\sin^2\theta)^{5/2}}\, d\theta \\
&= \int \frac{3\cos\theta}{3^5\cos^5\theta}\, d\theta \\
&= \frac{1}{3^4}\int \sec^4\theta\, d\theta
\end{aligned}
$$

Let $u = \tan\theta$. Then $du = \sec^2\theta\, d\theta$. Recall that $\sec^2\theta = 1 + \tan^2\theta$.

$$\frac{1}{3^4}\int \sec^4\theta\, d\theta = \frac{1}{3^4}\int (1+\tan^2\theta)\sec^2\theta\, d\theta$$

$$
\begin{aligned}
&= \frac{1}{3^4}\int 1+u^2\,du \\
&= \frac{1}{3^4}\left(u+\frac{u^3}{3}\right)+c \\
&= \frac{1}{3^4}\left(\tan\theta+\frac{\tan^3\theta}{3}\right)+c
\end{aligned}
$$

We know that $\frac{x+2}{3}=\sin\theta$. Sketch a right triangle whose opposite side has length $x+2$ and whose hypotenuse has length 3. Applying the Pythagorean theorem, we see that $\tan\theta=\dfrac{x+2}{\sqrt{5-4x-x^2}}$. Therefore,

$$
\boxed{\int \frac{1}{(5-4x-x^2)^{5/2}}\,dx = \frac{1}{3^4}\left(\frac{x+2}{\sqrt{5-4x-x^2}}+\frac{(x+2)^3}{3(5-4x-x^2)^{3/2}}\right)+c.}
$$

5. (a) Set up (but do not solve) the integral for the arc length along the curve $x=y+y^3$ from $y=1$ to $y=4$.

Parametrize the curve: $y=t$, $x=t+t^3$.

$$
\begin{aligned}
\text{Arc Length} = \int dS &= \int_{t=1}^{4}\sqrt{(x'(t))^2+(y'(t))^2}\,dt \\
&= \int_1^4 \sqrt{(1+3t^2)^2+1}\,dt.
\end{aligned}
$$

(b) Set up (but do not solve) the integral for the surface area of the surface obtained by rotating the curve given by

$$
x=a\cos^3 t,\quad y=a\sin^3 t,\quad 0\leqslant t\leqslant \pi/2
$$

about the x-axis. Here a is an arbitrary constant.

$$
\begin{aligned}
\text{Surface Area} &= 2\pi\int_{t=0}^{\pi/2}|a|\sin^3 t\sqrt{(x'(t))^2+(y'(t))^2}\,dt \\
&= 2\pi\int_0^{\pi/2}|a|\sin^3 t\sqrt{(3a\cos^2(t)\sin(t))^2+(3a\sin^2(t)\cos(t))^2}\,dt \\
&= 2\pi\int_0^{\pi/2}3a^2\sin^3 t\sqrt{\sin^2 t\cos^2 t(\cos^2 t+\sin^2 t)}\,dt \\
&= 6a^2\pi\int_0^{\pi/2}\sin^4 t\cos t\,dt.
\end{aligned}
$$