Example: Length of a Parabola

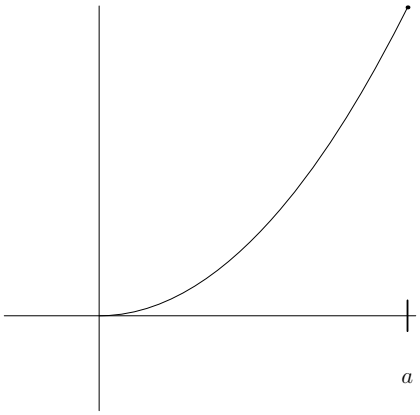

Figure 1: Arc length of $y = x^2$ over $0 \leq x \leq a$.

To find the arc length of a parabola we start with:

$$\begin{aligned} y &= x^2 \\ y' &= 2x \\ ds &= \sqrt{1 + (2x)^2}\, dx \\ &= \sqrt{1 + 4x^2}\, dx. \end{aligned}$$

So the arc length of the parabola over the interval $0 \leq x \leq a$ is:

$$\int_0^a \sqrt{1 + 4x^2}\, dx.$$

This is the answer to the question, but it would be more useful to us if we could write it in a simpler form. That's why we studied techniques of integration. To evaluate this integral we use the following trig substitution:

$$\begin{aligned} x &= \frac{1}{2} \tan u \\ dx &= \frac{1}{2} \sec^2 u \end{aligned}$$

When we do, we find that:

$$\int_0^a \sqrt{1 + 4x^2}\, dx = \left[\frac{1}{4} \ln(2x + \sqrt{1 + 4x^2}) + \frac{1}{2} x \sqrt{1 + 4x^2} \right]_0^a$$

(you may have seen parts of this calculation in a recitation video).

18.01SC Single Variable Calculus
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