

Welcome back to recitation. In this video I want to talk to you about another test for convergence we have for series, that you haven't really spent any time looking at this particular one. And it's pretty helpful, and also will help us understand something about Taylor series, which I'll do in another video.

So this is the ratio test. And you'll understand the name of this test, momentarily. So we're going to start with a series-- sorry-- we're going to start with a series that we'll just say, we'll call each term a sub n . And I'm not going to tell you where n starts, because it doesn't matter. It's really going to only matter what's happening out at infinity. And to make things simpler, we're going to let all the terms be positive. OK?

You, if they're not positive, you can take the absolute value of all the terms and still make some conclusions in terms of absolute convergence. So that's just a little sidebar. But let's just deal with all the terms positive, so we don't have to worry about anything.

And now, what does the ratio test say? Well, the ratio test says that first, we consider a certain ratio. The limit as n goes to infinity of a sub n plus 1 over a sub n . OK?

So we consider this limit. Well, that's your ratio. What is this doing? This is taking a term, and it's dividing by the previous term. You do that for all of the values for n , as n goes to infinity, and you look at the limit, if it exists. OK? If the limit doesn't exist, then you can't use this test. So sometimes that will happen. But if the limit exists, you can use this test, and you say it equals L .

And then you have the following conclusions you can make. So here are the conclusions. There's three of them. So if L is less than 1, then the series converges. OK? That's nice. That's good. If L is bigger than 1, the series diverges. OK? That's another good thing. And then the last one is, if L equals 1, you can't conclude anything. So I will try and convince you of that fact with a few examples later.

But let's look at this. So you look at the ratio, and if the ratio is less than 1, then you actually can conclude the series converges. And if the ratio is bigger than 1, you can conclude that the series diverges. So let me just give you a little understanding of why this one is true, and then the same kind of logic can be used for this second one. So we'll try and understand just at least a little bit why, when L is less than 1, the series converges.

OK. So let me just start writing here. So if L is less than 1, then this means that we have that a sub n plus 1 over a sub n as n goes to infinity is equal to L , which is less than 1. Right? So we can pick something between L and 1. We can pick a number between 1 and 1, because L is strictly less than 1.

And so what I'm going to do, is I'm going to say, I'm going to call this thing r . Some number between L and 1. OK?

So what does that mean? That means that for large n , we have a_{n+1} over a_n -- sorry, that looks like I'm adding 1 to the a_{n+1} -- that's a subscript-- a_{n+1} over a_n is less than some fixed r . So I'm picking a value r between L and 1 . And then this is true for all large N . And when you're doing math, sometimes people say, for n , you know, bigger than or equal to some fixed value. Basically, if you go far enough out in the sequence, then all the values bigger than some fixed N have this ratio. OK? But let's get fancy with this. This we can rewrite as r^{n+1} over r^n . Right? This is just r .

And now if I do a little moving around, what do I see? I see that a_{n+1} over r^{n+1} is less than a_n over r^n .

Now, this might be weird. What did I do? I just multiplied through by the a_n , and I divided by the r^{n+1} . I get this thing, and then I see that this ratio, as n goes to infinity, the ratio between a_n and r^n is decreasing. It's a decreasing, because the next term, it's smaller. Right? And so the point is that if I go far enough out, if I start, say, past this N , if I go far enough out, then I always have that a_n is less than some constant times r^n . OK? So this means-- this implies-- a_n is always less than some constant times r^n . Right?

And now, what do we do? We do our comparison test. OK? We do our comparison test. This is what's going to tell us that the series converges. Now, again, this isn't necessarily true all the way through the series, but it's true when you're far enough out, after this N . And if a series converges at the end, the beginning is just a finite sum. So we don't have to worry about what's going on at the beginning.

So again, we're at this place. We want to know what happens to the sum of a_n , of all the terms a_n . Well, we know that's going to be less than k times the sum r^n . Right? Because each a_n is less than some constant times r^n .

Now why does this converge? What do we know about r ? r we chose, we said it's between L and 1 . In particular, it's less than 1 . This is a geometric series. Geometric series, when r is less than 1 , we know it converges. And so this one converges, so then this one converges.

So that's the logic behind it. We're going to now-- you know, we're going to now use it. But I want to point out that if you liked that, you can come back over here, and you can do the same kind of reasoning for why, if L is bigger than 1 , a_n , the series, the sum of the a_n 's diverges. And that's going to come down to, now you choose an r that's between L and 1 , but it has to be bigger than 1 . OK? And then you can look at that. Or maybe r has to be bigger than L . I didn't even work that one out all the way. But you can do it. You put an r in there somewhere. And the same kind of logic, because the r will be bigger than one, you're going to get to a place where probably the inequality sign is going to go the opposite way, right? And then you'll have a series that

diverges, and the other one will be bigger than that one. And so that's how the logic is going to work.

So you have to figure out where the r goes, but I guarantee you'll want the r bigger than 1. And then you can, you'll have to have the inequality signs be opposite what they are here. OK? You'll see, they're going to be opposite there.

OK. Now let's get some examples. Example 1. Let's look at some that we know, and then let's look at some that we don't know. OK? So let's look for example first at $1/n$. Alright? Let's use the ratio test on $1/n$. Maybe this seems funny, 'cause what do we know about it? We know it diverges, right? But let's check, if this tells us. The limit of n goes to infinity of-- well, what's the n plus first term of this? It's going to be $1/(n+1)$. And what's the n th term of this? It's going to be $1/n$. And so we get, it's the limit as n goes to infinity of $n/n+1$, and that equals 1. Hmm. So this one didn't work. This one didn't tell us anything. And, OK. But we know this one diverges. So we know that-- this makes us think, well, maybe when L is 1, then we know it diverges. But just to make sure we don't make that conclusion, we don't draw that conclusion, let's look at $1/n^2$. And what are the terms there? $1/(n+1)$ and $1/n$. The limit as n goes to infinity. Well, the n plus first term is going to be $1/(n+1)^2$, and the n th term is going to be $1/n^2$, which is going to be the limit as n goes to infinity of $n^2/(n+1)^2$, which is also equal to 1. And what do we know about this one? This one converges. So this one gave us the L is equal to 1, but we know this one diverges. And this one gave us L is equal to 1, and we know it converges. So we know that when L equals 1, we really cannot conclude convergence or divergence. OK? L equals 1 doesn't let us draw any conclusions.

But now let's see something where, you know, we can draw a conclusion, because it would be no fun if this test never told us anything. It probably wouldn't be a test, then.

So let's try this one. Let's try $4^n/n \times 3^n$. Let's see what that one does. All right? So let's see. What is, we need the limit as n goes to infinity. We need the n plus first term, so let's plug in $n+1$ for all of these. And I'm actually going to do a little trick here, and I'll explain it as I go. $4^{n+1}/(n+1) \times 3^{n+1}$ to the $n+1$. I'm going to multiply by $1/n$. Because sometimes that's a lot easier to do. I could have done it on these other ones, maybe, but now I'm going to do it on this one.

So this is actually what the a_{n+1} is going to look like, right? I had to put in $n+1$ to get a_{n+1} . This is a_n . I'm going to write down $1/n$, and that's going to give me $n \times 3^n / 4^n$.

And now let's start simplifying. I have $3^n / 3^{n+1}$. I'm left with just a 3 there. 4^n and 4^{n+1} . I'm left with just a 4 there. And now the limit as n goes to infinity-- let's just rewrite it so I know what it is. Let's see. I have $4^n / 3^{n+1}$. Well, the 4 and the 3 I can actually just pull out. But what did I have here? $n/n+1$? The limit of n goes to infinity of that, that equals to $4/3$. That's bigger than 1. So that

actually diverges, OK?

And I have one more example, and I'm almost out of space. And let me actually come over here and figure out what example I wanted. Ah. Sorry about that. I knew I had one more. OK. n to the tenth over 10 to the n . So it's kind of interesting one, because you have, you have an exponential and then you have a power of n . So let's look at this one. So we need to consider limit as n goes to infinity. So I put in $n + 1$ first. $n + 1$ to the tenth over 10 to the $n + 1$. And then I'm going to just, remember, do times 1 over a sub n . So I'm going to have 10 to the n over n to the tenth. So again, I took a sub n , I did 1 over that, that's just the reciprocal. And now let's start dividing, if I'm allowed to divide. Yeah, I've got 10 to the n there, and 10 to the $n + 1$ there, so that gives me a single 10 in the denominator. And so now I really have the limit as n goes to infinity of 1 over 10 times $n + 1$ to the tenth over n to the tenth. Well, that's equal to-- $n + 1$ to the tenth over n to the tenth,

you might start to get nervous and think, "Oh my gosh! These powers are getting really big! It might make a difference that that 1 is there." It doesn't make a difference that that 1 is there, because if you actually expand this out, the leading term is just n to the tenth. And we know that the highest order, or the highest degree is going to win out, so it's going to be n to the tenth over n to the tenth is how it's going to behave in the limit. So this part's just going to go to 1 , so I get $1/10$. Oh, that's less than 1 ! Yay! So this series converges.

OK? So we had one that diverges, one that converges, and a few where we couldn't get conclusions by this test. And one point I want to make about this, is that in some cases, you have the integral test already, and sometimes that's easy and that helps you. In the case of these examples where we couldn't tell where I was equal to 1 , the integral test is going to tell you something. But in this case, it's a little bit harder to deal with this, as an integral test. You can still do it, but it's a little bit harder. And so this, maybe, is a little bit quicker way to deal with these types of, types of problems.

And the big thing we're going to do, is in the next video on the ratio test, I'm going to show you how you can use this to determine the radius of convergence for these Taylor series. So that's actually going to be kind of exciting, and that'll be in our next video.