33. Gauss Theorem

Theorem 33.1 (Gauss' Theorem)**.** *Let $M \subset \mathbb{R}^3$ be a smooth 3-manifold with boundary, and let $\vec{F}\colon M \longrightarrow \mathbb{R}^3$ be a smooth vector field with compact support.*

Then

$$\iiint_M \operatorname{div} \vec{F} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z = \iint_{\partial M} \vec{F} \cdot \mathrm{d}\vec{S},$$

where ∂M is given the outward orientation.

Example 33.2. *Three point charges are located at the points P_1, P_2 and P_3. There is an electric field*

$$\vec{E}\colon \mathbb{R}^3 \setminus \{P_1, P_2, P_3\} \longrightarrow \mathbb{R}^3,$$

which satisfies $\operatorname{div} \vec{E} = 0$.

Suppose there are four closed surfaces S_1, S_2, S_3 and S_4. Each S_i divides $\mathbb{R}^3$ into two pieces, which we will informally call the inside and the outside. S_1 and S_2 and S_3 are completely contained in the inside of S_4. The inside of S_1 contains the point P_1 but neither P_2 nor P_3, the inside of S_2 contains the point P_2 but neither P_1 nor P_3, and the inside of S_3 contains the point P_3 but neither P_1 nor P_2. The inside of S_4, together with S_4, minus the inside of S_1, S_2 and S_3 is a smooth 3-manifold with boundary. We have

$$\partial M = S'_1 \amalg S'_2 \amalg S'_3 \amalg S_4.$$

Recall that primes denote the reverse orientation. (33.1) *implies that*

$$\begin{aligned}
&\iint_{S_4} \vec{E} \cdot \mathrm{d}\vec{S} - \iint_{S_1} \vec{E} \cdot \mathrm{d}\vec{S} - \iint_{S_2} \vec{E} \cdot \mathrm{d}\vec{S} - \iint_{S_3} \vec{E} \cdot \mathrm{d}\vec{S} \\
&= \iint_{S_4} \vec{E} \cdot \mathrm{d}\vec{S} + \iint_{S'_1} \vec{E} \cdot \mathrm{d}\vec{S} + \iint_{S'_2} \vec{E} \cdot \mathrm{d}\vec{S} + \iint_{S'_3} \vec{E} \cdot \mathrm{d}\vec{S} \\
&= \iint_{\partial M} \vec{E} \cdot \mathrm{d}\vec{S} \\
&= \iint_M \operatorname{div} \vec{E} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z \\
&= 0.
\end{aligned}$$

In other words, we have

$$\iint_{S_4} \vec{E} \cdot \mathrm{d}\vec{S} = \iint_{S_1} \vec{E} \cdot \mathrm{d}\vec{S} + \iint_{S_2} \vec{E} \cdot \mathrm{d}\vec{S} + \iint_{S_3} \vec{E} \cdot \mathrm{d}\vec{S}.$$

Proof of (33.1). The proof (as usual) is divided into three steps.

Step 1: We first suppose that $M = \mathbb{H}^3$, upper half space. Suppose that we are given a vector field $\vec{G}\colon \mathbb{H}^3 \longrightarrow \mathbb{R}^3$, which is zero outside some box

$$K = [-a/2, a/2] \times [-b/2, b/2] \times [0, c/2].$$

We calculate:

$$\begin{aligned}
\iiint_{\mathbb{H}^3} \operatorname{div} \vec{G} \, \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}w &= \int_0^c \int_{-b}^b \int_{-a}^a \left(\frac{\partial G_1}{\partial u} + \frac{\partial G_2}{\partial v} + \frac{\partial G_1}{\partial w} \right) \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}w \\
&= \int_0^c \int_{-b}^b (G_1(a, v, w) - G_1(-a, v, w)) \, \mathrm{d}v \, \mathrm{d}w \\
&+ \int_0^c \int_{-a}^a (G_2(u, b, w) - G_2(u, -b, w)) \, \mathrm{d}u \, \mathrm{d}w \\
&+ \int_{-b}^b \int_{-a}^a (G_3(u, v, c) - G_3(u, v, 0)) \, \mathrm{d}u \, \mathrm{d}w \\
&= -\int_{-b}^b \int_{-a}^a G_3(u, v, 0) \, \mathrm{d}u \, \mathrm{d}w.
\end{aligned}$$

On the other hand, let's parametrise the boundary $\partial \mathbb{H}^3$, by

$$\vec{g}\colon \mathbb{R}^2 \longrightarrow \partial \mathbb{H}^3,$$

where

$$\vec{g}(u, v) = (u, v, 0).$$

In this case

$$\frac{\partial \vec{g}}{\partial u} \times \frac{\partial \vec{g}}{\partial v} = \hat{\imath} \times \hat{\jmath} = \hat{k}.$$

It follows that

$$\begin{aligned}
\iint_{(\partial \mathbb{H}^2)'} \vec{G} \cdot \mathrm{d}\vec{S} &= \iint_{\mathbb{R}^2} \vec{G} \cdot \hat{k} \, \mathrm{d}u \, \mathrm{d}v \\
&= \int_{-b}^b \int_{-a}^a G_3(u, v, 0) \, \mathrm{d}u \, \mathrm{d}v.
\end{aligned}$$

Therefore

$$\begin{aligned}
\iint_{\partial \mathbb{H}^2} \vec{G} \cdot \mathrm{d}\vec{S} &= \iint_{(\partial \mathbb{H}^2)'} \vec{G} \cdot \mathrm{d}\vec{S} \\
&= -\int_{-b}^b \int_{-a}^a G_3(u, v, 0) \, \mathrm{d}u \, \mathrm{d}v.
\end{aligned}$$

Putting all of this together, we have

$$\iiint_{\mathbb{H}^3} \operatorname{div} \vec{G} \, \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}w = \iint_{\partial \mathbb{H}^2} \vec{G} \cdot \mathrm{d}\vec{S}.$$

This completes step 1.

Step 2: We suppose that there is a compact subset $K \subset M$ and a parametrisation

$$\vec{g}\colon \mathbb{H}^3 \cap U \longrightarrow M \cap W,$$

such that

(1) $\vec{F}(\vec{x}) = \vec{0}$ for any $\vec{x} \in M \setminus K$.
(2) $K \subset M \cap W$.

We may write

$$\vec{g}(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w)).$$

Define

$$\vec{G}\colon \mathbb{H}^3 \longrightarrow \mathbb{R}^3,$$

by

$$\begin{aligned} G_1 &= F_1 \frac{\partial(y,z)}{\partial(v,w)} - F_2 \frac{\partial(x,z)}{\partial(v,w)} + F_3 \frac{\partial(x,y)}{\partial(v,w)} \\ G_2 &= -F_1 \frac{\partial(y,z)}{\partial(u,w)} + F_2 \frac{\partial(x,z)}{\partial(u,w)} - F_3 \frac{\partial(x,y)}{\partial(u,w)} \\ G_3 &= F_1 \frac{\partial(y,z)}{\partial(u,v)} - F_2 \frac{\partial(x,z)}{\partial(u,v)} + F_3 \frac{\partial(x,y)}{\partial(u,v)}, \end{aligned}$$

for any $(u, v, w) \in V$ and otherwise zero. Put differently,

$$\vec{G}(u, v, w) = \begin{cases} \vec{F} \cdot A & \text{if } (u, v, w) \in U \\ \vec{0} & \text{otherwise,} \end{cases}$$

where A is the matrix of cofactors of the derivative $D\vec{g}$.

One can check (that is, there is a somewhat long and involved calculation, similar, but much worse, than ones that appear in the proof of Green's Theorem or Stokes' Theorem) that

$$\begin{aligned} \operatorname{div} \vec{G} &= \operatorname{div} \vec{F} \det D\vec{g} \\ &= \operatorname{div} \vec{F} \frac{\partial(x,y,z)}{\partial(u,v,w)}. \end{aligned}$$

We have

$$\begin{aligned} \iiint_M \operatorname{div} \vec{F} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z &= \iiint_{\mathbb{H}^3} \operatorname{div} \vec{G} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\ &= \iint_{\partial \mathbb{H}^2} \vec{G} \cdot \mathrm{d}\vec{S}, \\ &= \iint_{\partial M} \vec{F} \cdot \mathrm{d}\vec{S}, \end{aligned}$$

where the last equality needs to be checked (this is relatively straightforward).

This completes step 2.

Step 3: We finish off in the standard way. We may find a partition of unity

$$1 = \sum_{i=1}^{k} \rho_i,$$

where ρ_i is a smooth function which is zero outside a compact subset K_i such that $\vec{F}_i = \rho_i \vec{F}$ is a smooth vector field, which satisfies the hypothesis of step 2, for each $1 \leq i \leq k$. We have

$$\vec{F} = \sum_{i=1}^{k} \vec{F}_i.$$

and so

$$\begin{aligned}
\iint_S \operatorname{curl} \vec{F} \cdot \mathrm{d}\vec{S} &= \sum_{i=1}^{k} \iint_S \operatorname{curl} \vec{F}_i \cdot \mathrm{d}\vec{S} \\
&= \sum_{i=1}^{k} \int_{\partial M} \vec{F}_i \cdot \mathrm{d}\vec{s} \\
&= \int_{\partial M} \vec{F} \cdot \mathrm{d}\vec{s}. \qquad \square
\end{aligned}$$