Equation of a plane

1. Find the equation of the plane containing the three points $P_1 = (1,0,1)$, $P_2 = (0,1,1)$, $P_3 = (1,1,0)$.

Answer: This problem is identical (with changed numbers) to the worked example we just saw.

The vectors $\overrightarrow{\mathbf{P_1P_2}}$ and $\overrightarrow{\mathbf{P_1P_3}}$ are in the plane, so

$$\mathbf{N} = \overrightarrow{\mathbf{P_1P_2}} \times \overrightarrow{\mathbf{P_1P_3}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{i}(-1) - \mathbf{j}(1) + \mathbf{k}(-1) = \langle -1, -1, -1 \rangle.$$

is orthogonal to the plane.

Now for any point $P = (x,y,z)$ in the plane, the vector $\overrightarrow{\mathbf{P_1P}}$ is also in the plane and is therefore orthogonal to $\mathbf{N}$. Expressing this with the dot product we get

$$\begin{aligned} & \mathbf{N} \cdot \overrightarrow{\mathbf{P_1P}} = 0 \\ \Leftrightarrow\quad & \langle -1,-1,-1 \rangle \cdot \langle x-1, y, z-1 \rangle = 0 \\ \Leftrightarrow\quad & -(x-1) - y - (z-1) = 0 \\ \Leftrightarrow\quad & x + y + z = 2. \end{aligned}$$