Parametric Curves

General parametric equations

We have seen parametric equations for lines. Now we will look at parametric equations of more general trajectories. Repeating what was said earlier, a parametric curve is simply the idea that a point moving in the space traces out a path.

We can use a parameter to describe this motion. Quite often we will use t as the parameter and think of it as time. Since the position of the point depends on t we write

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

to indicate that x, y and z are functions of t. We call t the parameter and the equations for x, y and z are called *parametric equations.*

It is not always necessary to think of the parameter as representing time. We will see cases where it is more convenient to express the position as a function of some other variable.

The position vector

In order to use vector techniques we define the *position vector*

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} = \langle x(t), y(t), z(t) \rangle.$$

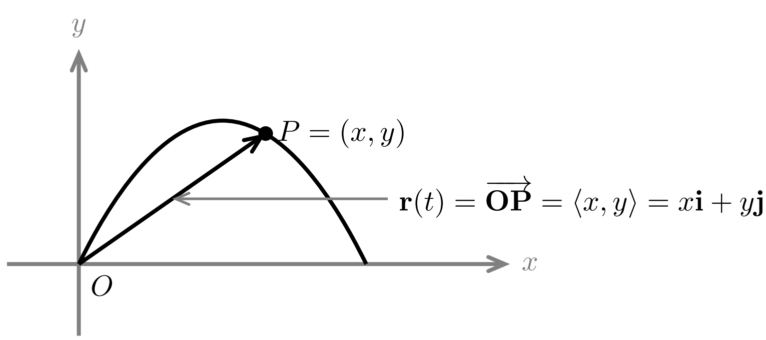
This is just the vector from the origin to the moving point. As the point moves so does the position vector –see the figure with example 1.

Example 1: Thomas Pynchon fires a rocket from the origin. Its initial x-velocity is $v_{0,x}$ and its initial y-velocity is $v_{0,y}$.

You've probably seen this, but in any case, physics tells us that the parametric equations for its parabolic trajectory are

$$x(t) = v_{0,x}t, \quad y(t) = -\frac{1}{2}gt^2 + v_{0,y}t.$$

At time t the rocket is at point $P = (x(t), y(t))$. The position vector can be written in many different ways: $\mathbf{r}(t) = \overrightarrow{\mathbf{OP}} = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} = \langle x, y \rangle$.

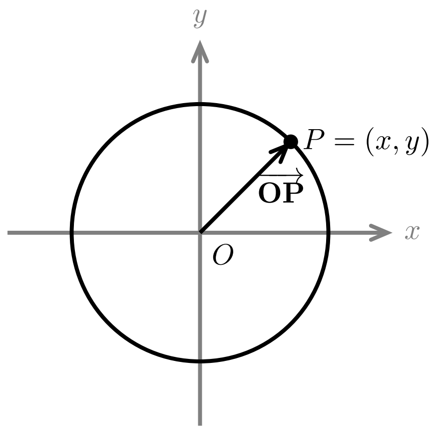


Next we will give a series of examples of parametrized curves. The most important are circles and lines. The last one is the *cycloid.* It is an important example which combines lines and circles.

Circles and ellipses
Consider the parametric curve in the plane

$$x(t) = a\cos t, \quad y(t) = a\sin t.$$

Easily we get the relation $x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2$. Therefore the trajectory is on a circle of radius a centered at O.



We will call $x(t) = a\cos t$, $y(t) = a\sin t$ the *parametric form* of the curve and $x^2 + y^2 = a^2$ the *symmetric form.*

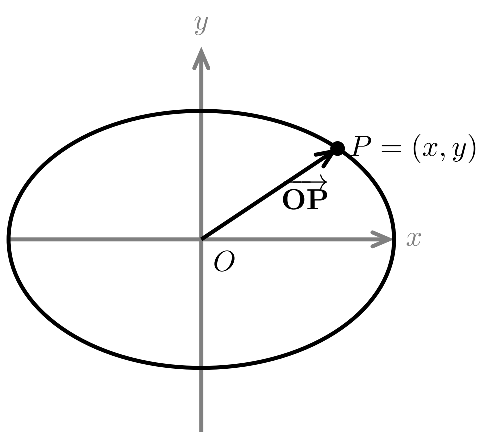
Note, a different parametrization, say

$$x(t) = a\cos(3t), \quad y(t) = a\sin(3t)$$

results in the same path, i.e. the circle $x^2 + y^2 = a^2$, but the two trajectories differ by how fast they travel around the circle.

The circle is easily changed to an ellipse by

$$\begin{aligned} &\text{parametric form:} \quad x(t) = a\cos t, \quad y(t) = b\cos t \\ &\text{symmetric form:} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \end{aligned}$$



Lines

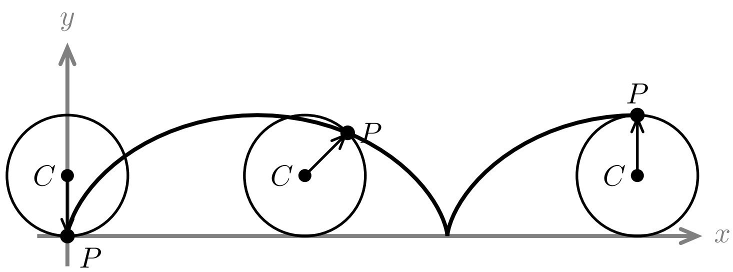
We review parametric equations of lines by writing the the equation of a general line in the plane. We know we can parametrize the line through (x_0, y_0) parallel to $\langle b_1, b_2 \rangle$ by

$$x(t) = x_0 + tb_1, \quad y(t) = y_0 + tb_2 \;\Leftrightarrow\; \mathbf{r}(t) = \langle x, y \rangle = \langle x_0 + tb_1, y_0 + tb_2 \rangle = \langle x_0, y_0 \rangle + t\langle b_1, b_2 \rangle.$$

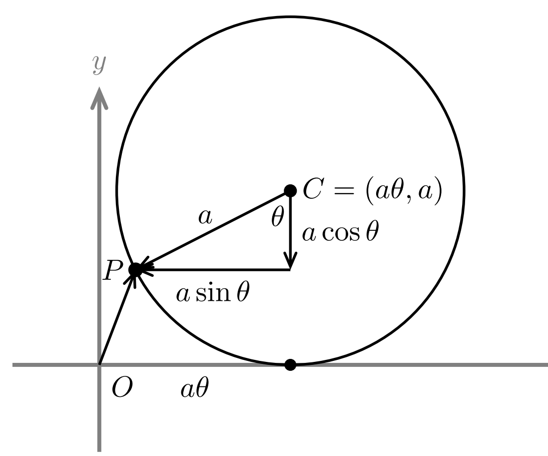
The cycloid

The cycloid has a long and storied history and comes up surprisingly often in physical problems. For us it is a curve that has no simple symmetric form, so we will only work with it in its parametric form.

The cycloid is the trajectory of a point on a circle that is rolling without slipping along the x-axis. To be specific, we'll follow the point P that starts at the origin.



The natural parameter to use is the angle θ that the wheel has turned. We'll use vector methods to find the position vector for P as a function of θ.



Our strategy is to break the motion up into translation of the center and rotation about the center. The figure shows the wheel after it has turned through a small θ. We see the position vector

$$\overrightarrow{\mathbf{OP}} = \overrightarrow{\mathbf{OC}} + \overrightarrow{\mathbf{CP}}.$$

We'll compute each piece separately.

After turning θ radians the wheel has rolled a distance $a\theta$, so the center of the circle is at $(a\theta, a)$, i.e.,

$$\overrightarrow{\mathbf{OC}} = \langle a\theta, a \rangle.$$

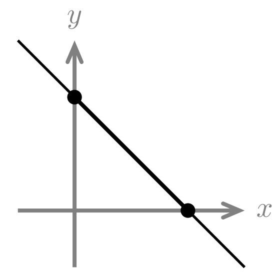
The figure also shows that

$$\overrightarrow{\mathbf{CP}} = \langle -a\sin\theta, -a\cos\theta \rangle.$$

Putting the pieces together we get parametric equations for the cycloid

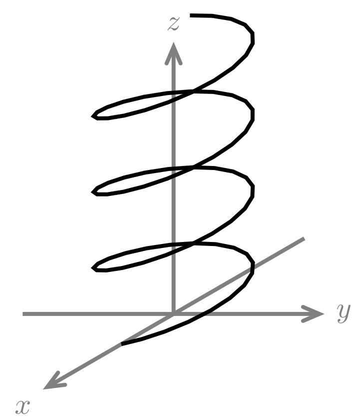
$$\overrightarrow{\mathbf{OP}} = \langle a\theta - a\sin\theta, a - a\cos\theta \rangle$$
$$\Leftrightarrow \quad x(\theta) = a\theta - a\sin\theta, \quad y(\theta) = a - a\cos\theta.$$

Example 2: (Where the symmetric form loses information.)
Find the symmetric form for $x = 3\cos^2 t,\ y = 3\sin^2 t$.



Easily we get: $x + y = 3$, with x, y non-negative.
The symmetric form shows a line, but the parametric trajectory only traces out a part of the line. In fact, it goes back an forth over the part of the line in the first quadrant.

Example 3: The curve $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + at\,\mathbf{k}$ is a helix winding around the z-axis.



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