Verify Green's Theorem in Normal Form

Verify that $\oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\,dA$ when $\mathbf{F} = x\hat{\mathbf{i}} + x\hat{\mathbf{j}}$ and C is the square with vertices $(0,0)$, (1,0), $(1,1)$ and $(0,1)$.

Answer:

Right hand side: Here $M = N = x$, so $\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\,dA = \iint_R 1\,dA = 1$.

Left hand side: $\oint_C M\,dy - N\,dx = \oint_C x\,dy - x\,dx$. We evaluate this line integral in four parts.

- $(0,0)$ to $(1,0)$.
$$\int_{x=0}^{x=1} x\cdot 0 - x\,dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}.$$
- $(1,0)$ to $(1,1)$.
$$\int_{y=0}^{y=1} 1\,dy - 1\cdot 0 = 1.$$
- $(1,1)$ to $(0,1)$.
$$\int_{x=1}^{x=0} x\cdot 0 - x\,dx = -\frac{1}{2}.$$
- $(0,1)$ to $(0,0)$.
$$\int_{y=1}^{y=0} 0\,dy - 0\cdot 0 = 0.$$

Since the sum of the line integrals along the components of C is 1, $\oint_C x\,dy - x\,dx = 1$. This confirms that the normal form of Green's Theorem is true in this example.