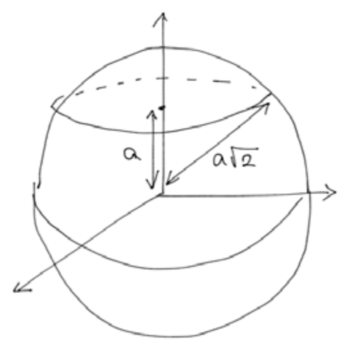
Problems: Spherical Coordinates

1. Find the volume of a solid spherical cap obtained by slicing a solid sphere of radius $a\sqrt{2}$ by a plane at a distance a from the center. (See picture.)



Answer: In session 76 we found the limits:

inner ρ: $a/\cos\phi$ to $a\sqrt{2}$,

middle ϕ: 0 to $\pi/4$,

outer θ: 0 to 2π.

$$\text{Volume} = \int_0^{2\pi}\int_0^{\pi/4}\int_{a/\cos\phi}^{a\sqrt{2}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

Inner: $\frac{1}{3}\rho^3 \sin\phi \Big|_{a/\cos\phi}^{a\sqrt{2}} = \frac{2a^3\sqrt{2}}{3}\sin\phi - \frac{a^3 \sin\phi}{3\cos^3\phi}.$

Middle: $\left[-\frac{2a^3\sqrt{2}}{3}\cos\phi - \frac{a^3}{6\cos^2\phi}\right]_{\phi=0}^{\pi/4} = -\frac{2a^3}{3} - \frac{a^3}{3} - \left(-\frac{2\sqrt{2}a^3}{3} - \frac{a^3}{6}\right) = \frac{2\sqrt{2}a^3}{3} - \frac{5a^3}{6}.$

Outer: $2\pi\left(\frac{2\sqrt{2}a^3}{3} - \frac{5a^3}{6}\right) = \frac{a^3\pi}{3}(4\sqrt{2} - 5) \approx 0.7a^3.$

The volume of the entire sphere is about $12a^3$ and we're looking at approximately the top sixth of its height. The sphere has more volume near its midpoint than at its top and bottom, so this answer seems reasonable.