18.03 Problem Set 4: Part II Solutions

Part I points: 13. 3, **14.** 8, **15.** 5, **16.** 4.

13. (a) [4] $\dot{z} + 2z = e^{(3+4i)t}$ has solution $z_p = e^{(3+4i)t}/((3+4i)+2) = \frac{5+4i}{25+16}e^{3t}(\cos(4t) + i\sin(4t))$ so $x_p = \mathrm{Re}(z_p) = e^{3t}(\frac{5}{41}\cos(4t) + \frac{4}{41}\sin(4t))$. The homogeneous equation has general solution $x_h = ce^{-2t}$, so $x = e^{3t}(\frac{3}{25}\cos(4t) + \frac{4}{25}\sin(4t)) + ce^{-2t}$.

(b) [4] $\ddot{z} + \dot{z} + 2z = e^{it}$ has solution $z_p = e^{it}/p(i)$; $p(i) = -1 + i + 2 = 1 + i$ so $z_p = \frac{1}{1+i}e^{it} = \frac{1-i}{2}(\cos t + i\sin t)$ and $x_p = \frac{1}{2}\cos t + \frac{1}{2}\sin t$. We can also write $p(i) = 1 + i = \sqrt{2}e^{i\pi/4}$, so $z_p = \frac{1}{\sqrt{2}}e^{-i\pi/4}e^{it} = \frac{\sqrt{2}}{2}e^{i(t-\pi/4)}$ and $x_p = \mathrm{Re}(z_p) = \frac{\sqrt{2}}{2}\cos(t - \frac{\pi}{4})$. The characteristic polynomial is $p(s) = s^2 + s + 2 = (s + \frac{1}{2})^2 + \frac{7}{4}$ with roots $-\frac{1}{2} \pm \frac{\sqrt{7}}{2}$, so the homogeneous equation has general solution $x_h = e^{-t/2}\left(a\cos\left(\frac{\sqrt{7}}{2}t\right) + b\sin\left(\frac{\sqrt{7}}{2}t\right)\right)$ or $x_h = Ae^{-t/2}\cos\left(\frac{\sqrt{7}}{2}t - \phi\right)$. The general solution is $x = x_p + x_h$.

(c) [4] I measure the first six zeros to be $2.36, 4.76, 7.12, 9.48, 11.89, 14.19$. The successive differences are $2.40, 2.36, 2.36, 2.41, 2.30$, with an average of 2.366. My measured pseudoperiod is twice this, 4.73. The damped circular frequency of this system is $\omega_d = \frac{\sqrt{7}}{2}$ so the pseudoperiod is $\frac{2\pi}{\omega_d} = \frac{4\pi}{\sqrt{7}} \simeq 4.7496$. Not bad agreement.

(d) [4] I measure the amplitude as 0.71. It looks like $x_p(0) \simeq 0.50$. Computed amplitude is $\frac{\sqrt{2}}{2} \simeq 0.707$. Computed value is $x_p(0) = \frac{\sqrt{2}}{2}\cos(-\frac{\pi}{4}) = \frac{1}{2}$. Good agreement! $\dot{x}_p(t) = -\frac{\sqrt{2}}{2}\sin(t - \frac{\pi}{4})$ so $\dot{x}_p(0) = -\frac{\sqrt{2}}{2}\sin(-\frac{\pi}{4}) = \frac{1}{2}$.

(e) [5] We need to select x_h so that $x_h(0) = -x_p(0) = -\frac{1}{2}$ and $\dot{x}_h(0) = -\dot{x}_p(0) = -\frac{1}{2}$. It's more convenient to use the rectangular expression for x_h for this. $x_h(0) = a$ so $a = -\frac{1}{2}$. $\dot{x}_h = e^{-t/2}\left(\left(-\frac{1}{2}a + \frac{\sqrt{7}}{2}b\right)\cos\left(\frac{\sqrt{7}}{2}t\right) + (\cdots)\sin\left(\frac{\sqrt{7}}{2}t\right)\right)$ so $\dot{x}_h(0) = -\frac{1}{2}a + \frac{\sqrt{7}}{2}b$. Thus $b = \frac{2}{\sqrt{7}}\left(\dot{x}_h(0) + \frac{1}{2}a\right) = \frac{2}{\sqrt{7}}\left(-\frac{1}{2} - \frac{1}{4}\right) = -\frac{3}{2\sqrt{7}}$. $x_h = e^{-t/2}\left(-\frac{1}{2}\cos\left(\frac{\sqrt{7}}{2}t\right) - \frac{3}{2\sqrt{7}}\sin\left(\frac{\sqrt{7}}{2}t\right)\right)$.

14. (a) [8] Since the input signal has amplitude 1, the gain is the amplitude of the system response. The equation is $\ddot{x} + \frac{1}{2}\dot{x} + 4x = 4\cos(2t)$. The complex replacement is $\ddot{z} + \frac{1}{2}\dot{z} + 4z = 4e^{2it}$. Since $p(2i) = (2i)^2 + \frac{1}{2}(2i) + 4 = i$, $z_p = 4e^{2it}/i$, so $x_p = \mathrm{Re}(z_p) = 4\sin(2t)$: the gain is 4. Also the phase lag of the sine behind the cosine is $\phi = \frac{\pi}{2}$. The time lag is $t_0 = \frac{\phi}{\omega} = \frac{\pi}{4} \simeq 0.7854$.

(b) [8] $H(\omega) = \dfrac{4}{p(i\omega)} = \dfrac{4}{4 - \omega^2 + \frac{1}{2}i\omega}$. The gain is $|H(\omega)| = \dfrac{4}{\sqrt{(4-\omega^2)^2 + \frac{1}{4}\omega^2}}$.

The phase lag ϕ is $-\mathrm{Arg}(H(\omega)) = \mathrm{Arg}(p(i\omega))$ so $\tan\phi = \dfrac{\mathrm{Im}(p(i\omega))}{\mathrm{Re}(p(i\omega))} = \dfrac{\omega/2}{4 - \omega^2}$.

15. (a) [6] $p(s) = s + 1$ and $p(-1) = 0$, so we are in resonance. $p'(s) = 1$ so the ERF/Resonant gives $x_p = te^{-t}$.

(b) [6] We can't apply undetermined coefficients directly since $p(0) = 0$. Let $u = \dot{x}$, so $\ddot{u} - u = t^2 + 1$. Try $u = at^2 + bt + c$, so $\ddot{u} = 2a$ and $t^2 + 1 = \ddot{u} - u = -at^2 - bt + (2a - c)$ implies $a = -1$, $b = 0$, $2a - c = 1$ or $c = -3$: so $u_p = -t^2 - 3$. Then x_p is the integral of u_p: $x_p = -\frac{1}{3}t^3 - 3t$. To solve the homogeneous equation, factor $p(s) = s(s-1)(s+1)$ so $x_h = c_1 + c_2e^t + c_3e^{-t}$. $x = x_p + x_h$.