18.03 Class 29, April 16, 2010

Laplace Transform IV: The pole diagram

1. Another example
2. t-shift rule
3. Poles
4. What the pole diagram of F(s) says about f(t)

[1] We saw that if p(D)w = delta(t) ,

i.e. a_n w^(n) + ... + a_1 w' + a_0 w = delta(t)

with rest initial conditions, then p(s) W(s) = 1 ,

or W(s) = 1/p(s) , or w(t) = L^{-1}(1/p(s))

w(t) = weight function
W(s) = transfer function

This is another RULE.

Other input signals?

x" + 4x = cos(2t) rest initial conditions

(No delta functions in this signal, so we know that x(0+) = 0 and x'(0+) = 0, but that will come out automatically.)

X = W(s) LT[cos(2t)] = 2s/(s^2+4)^2

From our latest addition to the table, we find that

x = u(t) 1/2 t sin(t)

(Resonance!)

GENERAL FACT: p(D) x = f(t) with rest initial conditions

has Laplace transformed equation

p(s) X(s) = F(s)

with solution

X(s) = W(s) F(s)

In the s-domain, the system response is obtained by multiplying by the transfer function!

[2] Another rule: Let a > 0 and define

f_a(t) = 0 if t < a

= f(t-a) if t > a

(If f_s(t) contains c delta(t) , I also want f_a(t) to contain
c delta(t-a) . So for example delta_a(t) = delta(t-a) .)

If you know F(s) , what is the LT of f_a(t) ?

integral_{0-}^infty f_a(t) e^{-st} dt = integral_{a-}^\infty f(t-a) e^{-st} dt

Substitution: u = t-a , du = dt ,

... = integral_{0-}^\infty f(u) e^{-s(u+a)} du

= e^{-as} integral_{0-}^infty f(u) e^{-su} du

= e^{-as} F(s)

So we have the t-shift rule: f_a(t) -----> e^{-as} F(s)

Eg u_a(t) ----> e^{-as}/s   and   delta_a(t) ----> e^{as} , as we know.

[3] F(s) "essentially" determines f(t) , but as we have seen the path
from F(s) to f(t) can be difficult. But there are certain features
of f(t) which can be EASILY read off from F(s) . They have to do with
the long term behavior of f(t) .

Typical LTs are F(s) = 1/s or   F(s) = 1/(s^2+2s+5) .

F(s) is complicated because it assigns to each complex number s another
complex number F(s) . Let's content ourselves with understanding the
absolute value of F(s) : |F(s)| . This assigns a real number to each
complex number. In 18.02 you learned how to think of functions on the
plane. It has a graph which is a surface in 3-space.

For example |1/s| = 1/|s| = 1/(distance from the origin). The graph of
this function is radially symmetric: it is a surface of revolution of one
branch of the hyperbola z = 1/x . It rises towards infinity as s goes
to 0, and falls off to zero as s grows large.

The point s = 0 is called a "pole" of F(s) = 1/s . Its presence, and
location, is a simple gross feature of F(s) . In fact, it's the only
thing you see from a distance!

How about F(s) = 1/(s^2+2s+5) ? The poles occur at the roots of the
denominator: s^2+2s+5 = (s+1)^2 + 4 has roots -1 +- 2i .
This is a two-ring circus!

And the sum has three poles.

Have you figured out what functions have those two F's as LT's?

L[1] = 1/s   L[(1/2)e^{-t} sin(2t)] = 1/((s+1)^2 + 4) .

Region of convergence again:
The location of the poles explains the region of convergence.

F(s) = integral_{0-}^infty e^{-st} f(t) dt

Since |e^{-st}| depends only on Re(s) , the region of convergence will always be to the right of some line Re(s) = a , or empty, or the whole plane. The value of F(s) at a pole is infinite; this reflects the divergence of the improper integral.

The region of convergence is the half plane to the right of the rightmost pole.

[4] What the poles of F(s) tell us about f(t)

Knowing where the poles are is just a small part of knowing the whole of F(s), and it doesn't tell you everything about f(t) .

Two examples:

(1) If f(t) = 0 for t > A , then the improper integral isn't so improper -

F(s) = integral_{0-}^A e^{-st} f(t) dt

- so it converges for ALL s : no poles at all.

(2) e^{-as} is never zero and has no poles: so the poles of e^{-as}F(s) are the same as the poles of F(s) : time translation of f(t) doesn't affect the pole diagram of F(s) . So the pole diagram can't see phase.

But the pole diagram does say a lot about the long term behavior of f(t).

Example 1. f(t) = sin(2t). F(s) = 4/(s^2_4) has poles at 2i and -2i.
Any f(t) such that F(s) has this pole diagram exhibits (for large t)
(a) oscillation with circular frequency 2 , and
(b) neither exponential growth nor exponential decay.

[The Laplace transform of f(t) = t sin(2t) is 4s/((s^2+4)^2) , which has the same pole diagram. This function does grow with t , but less than exponentially.]

Example 2: f(t) = e^{-t} sin(2t) . F(s) = 2s/((s+1)^2+4) has poles at -1+2i and -1-2i . Any f(t) such that F(s) has this pole diagram exhibits (for large t)
(a) oscillation with circular frequency 2 , and
(b) exponential decay on the order of e^{-t} .

Example 3. f(t) = 1 + e^{-t} sin(2t) . F(t) = 1 + 2s/((s+1)^2+4) , which has poles at 0 , -1+2i, and -1-2i . Any f(t) such that F(s) has this pole diagram exhibits (for large t)

(a) sub-exponential growth/decay,
i.e. for any a > 0 , e^(-at) < |f(t)| < e^(at) .

(b) oscillation with circular frequency 2 but with amplitude decaying like e^{-t} .

The rightmost poles give the dominant information.

A major lesson: if all the poles of $F(s)$ have negative real part, then the function $f(t)$ decays to zero exponentially.

The position of the poles of $F(s)$ gives detailed information about the long-term rate of growth and oscillation of $f(t)$, information which it can be hard to glean from the graph of $f(t)$ ... all exponentials look rather similar ...

So when you solve an ODE using LT, it may be good enough to stop with $X(s)$. Find its poles, and you know what the solution behaves like in the long run.