18.03 Recitation 24, May 6, 2010

Matrix exponentials

1. On the trace-determinant plane, where can you guarantee that any matrix with this value of trace and determinant is stable? Unstable? Neutrally stable? Are there there any values of the trace and determinant for which there are matrices exhibiting more than one type of limiting behavior?

Stable solutions occur if and only if both eigenvalues have negative real part. This happens exactly when the trace is negative and the determinant is positive.

Unstable solutions occur when there is at least one eigenvalue with positive real part. This happens either when the determinant is negative, or when the trace is positive.

Neutrally stable solutions occur either when the trace is zero and the determinant is positive (giving periodic solutions), or when the determinant is zero and the trace is negative (giving combs).

When the trace and determinant are both zero, the zero matrix gives neutrally stable behavior, while the matrix that has exactly one nonzero entry, in the upper right, gives unstable behavior.

2. In this problem, $A = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}$ and we are interested in the equation $\dot{\mathbf{u}} = A\mathbf{u}$.

(a) Find a fundamental matrix $\Phi(t)$ for A.

The eigenvalues of A are solutions to $(1-\lambda)^2+4 = \lambda^2-2\lambda+5 = 0$, so we let $\lambda_1 = 1+2i$. Any eigenvector corresponding to this eigenvalue is killed by $\begin{bmatrix} -2i & 1 \\ -4 & -2i \end{bmatrix}$, so we set $v_1 = \begin{bmatrix} 1 \\ 2i \end{bmatrix}$. We have a solution

$$\begin{aligned} u(t) &= e^{(1+2i)t} \begin{bmatrix} 1 \\ 2i \end{bmatrix} \\ &= e^t(\cos 2t + i\sin 2t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2ie^t(\cos 2t + i \sin 2t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} e^t \cos 2t \\ -2e^t \sin 2t \end{bmatrix} + i \begin{bmatrix} e^t \sin 2t \\ 2e^t \cos 2t \end{bmatrix} \end{aligned}$$

By taking real and imaginary parts, we get a fundamental matrix:

$$\begin{bmatrix} e^t \cos 2t & e^t \sin 2t \\ -2e^t \sin 2t & 2e^t \cos 2t \end{bmatrix}$$

(b) Find the exponential matrix e^{At}.

When $t = 0$, the real part is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the imaginary part is $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$, so we divide the second column by two to get $\begin{bmatrix} e^t \cos 2t & 1/2e^t \sin 2t \\ -2e^t \sin 2t & e^t \cos 2t \end{bmatrix}$

(c) Find the solution to $\dot{\mathbf{u}} = A\mathbf{u}$ with $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is the sum of the real and imaginary parts of our solution $u(0)$ above, so we just take the sum: $\begin{bmatrix} e^t(\cos 2t + \sin 2t) \\ 2e^t(\cos 2t - \sin 2t) \end{bmatrix}$. In general, we would get our answer by multiplying $e^{At}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(d) Find a solution to $\dot{\mathbf{u}} = A\mathbf{u} + \begin{bmatrix} 5 \\ 10 \end{bmatrix}$. What is the general solution? What is the solution with $\mathbf{u}(0) = \mathbf{0}$?

We can find a constant solution by taking $u = -A^{-1}\begin{bmatrix} 5 \\ 10 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$. This is a solution, since we have $\dot{\mathbf{u}} = A\mathbf{u} + \begin{bmatrix} 5 \\ 10 \end{bmatrix} = -\begin{bmatrix} 5 \\ 10 \end{bmatrix} + \begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The general solution is given by $e^{At}\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} 1 \\ -6 \end{bmatrix}$, or $\begin{bmatrix} ae^t \cos 2t + b/2e^t \sin 2t + 1 \\ -2ae^t \sin 2t + be^t \cos 2t - 6 \end{bmatrix}$.

We plug in $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$ to get $\begin{bmatrix} -e^t \cos 2t + 3e^t \sin 2t + 1 \\ 2e^t \sin 2t + 6e^t \cos 2t - 6 \end{bmatrix}$.

3. Suppose $\mathbf{u_1}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (constant function) and $\mathbf{u_2}(t) = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$ are solutions to the equation $\dot{\mathbf{u}} = B\mathbf{u}$.

(a) What is the general solution? What is the solution $\mathbf{u}(t)$ with $\mathbf{u}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$? What is the solution with $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$?

The general solution is $u(t) = a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}$.

$\mathbf{u_1}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so $\mathbf{u}(t) = 2\mathbf{u_1}(t) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

$\mathbf{u_1}(0) + \mathbf{u_2}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, so $\mathbf{u}(t) = \begin{bmatrix} e^t/2 + 1/2 \\ -e^t/2 + 1/2 \end{bmatrix}$.

(b) Find a fundamental matrix, and compute the exponential e^{Bt}. What is e^B?

Since the two given solutions are linearly independent, a fundamental matrix is $\begin{bmatrix} 1 & e^t \\ 1 & -e^t \end{bmatrix}$.

Evaluating the matrix at $t = 0$ gives $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ which has inverse $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$.

Multiplying this by our fundamental matrix gives us:

$$\begin{aligned} e^{Bt} &= \begin{bmatrix} 1 & e^t \\ 1 & -e^t \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \\ &= \begin{bmatrix} e^t/2 + 1/2 & 1/2 - e^t/2 \\ 1/2 - e^t/2 & e^t/2 + 1/2 \end{bmatrix} \end{aligned}$$

Evaluating e^{Bt} at $t = 1$ gives us $e^B = \begin{bmatrix} (e+1)/2 & (1-e)/2 \\ (1-e)/2 & (e+1)/2 \end{bmatrix}$.

(c) What are the eigenvalues and eigenvectors of B?

The eigenvalues of e^B are 1 and e, with eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, respectively. The eigenvalues of B are the logarithms of the eigenvalues of e^B, so they are 0 and 1, with the same eigenvectors.

(d) What is B?

$B \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is zero, so B has the form $\begin{bmatrix} a & -a \\ c & -c \end{bmatrix}$. $B \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, so $a = 1/2$ and $c = -1/2$. Thus, $B = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$.