18.03SC Differential Equations, Fall 2011

Transcript – Lecture 6

I assume from high school you know how to add and multiply complex numbers using the relation i^2 = -1. I'm a little less certain that you remember how to divide them. I hope you read last night by way of preparation for that, but since that's something we're going to have to do a lot of a differential equations, so remember that the division is done by making use of the complex conjugate.

So, if z = a + bi, some people write a + ib, and sometimes I'll do that too if it's more convenient. Then, the complex conjugate is what you get by changing i to negative i. And, the important thing is that the product of those two is a real number. The product of these is (a^2 - (ib)^2), which makes a^2 + b^2 because i^2 = -1. So, the product of those, that's what you multiply if you want to multiply this by something to make it real. You always multiplied by its complex conjugate.

And that's the trick that underlines the doing of the division. So, for example, I better hang onto these or I'll never remember all the examples. Suppose, for example, we wanted to calculate (2 + i) / (1 - 3i). To calculate it means I want to do the division; I want to express the answer in the form a plus bi. What you do is multiply the top and bottom by the complex conjugate of the denominator in order to make it real. So, it's (1 + 3i) / (1 + 3i), as they taught you in elementary school, that is one, in a rather odd notation; therefore, multiplying doesn't change the value of the fraction. And so, the denominator now becomes 1^2 + 3^2 = 10.

And, the numerator is, learn to do this without multiplying out four terms. You must be able to do this in your head. And, you always do it by the grouping, or post office method, whatever you want to call it, namely, first put down the real part, which is made out of two times one minus three times one. So, that's negative one. And then, the imaginary part, which is i times one. That's one, coefficient one, plus 6i.

So, that makes 7i. Now, some people feel this still doesn't look right, if you wish, and for some places and differential equations, it will be useful to write that as -1/10 + 7/10i. And, now it's perfectly clear that it's in the form a plus bi. So, learn to do that if you don't know already. It's going to be important. Now, the main thing today is the polar representation, which sometimes they don't get to in high school. And if they do, it's usually not in a grown up-enough in a form for us to be able to use it. So, I have to worry about that little bit. The polar representation, of course, is nominally just the switch to polar coordinates. If here's a plus bi, then this is r, and that's theta.

And therefore, this can be written as, in the polar form, that would be r cos(theta) + i, or r cos(theta). That's the A part. And, the B part is, the imaginary part is r sin(theta)*i. Now, it would be customary, at this point, to put the i in front, just because it looks better. The complex numbers are commutative, satisfied to commutative law of multiplication, which means it doesn't matter in multiplication whether you put i in front or behind. It's still the same answer. So, this would be r cos(theta) + i*r sin(theta), which, of course, will factor out, and will make it cos(theta) + i sin(theta). Now, it was Euler who took the decisive step and said, hey, look, I'm going to call that e^(i theta).

Now, why did he do that? Because everything seemed to indicate that it should. But that's certainly worth the best color we have, which is what? We are getting low here. Okay, nonetheless, it's worth pink. I will even give him his due, Euler. Sometimes it's called Euler's formula, but it really shouldn't be. It's not a formula. It's a definition. So, in some sense, you can't argue with it. If you want to call putting a complex number in a power, and calling it that, you can. But, one can certainly ask why he did it. And the answer, I guess, is that all the evidence seemed to point to the fact that it was the thing to do.

Now, I think it's important to talk about a little bit because I think it's, in my opinion, if you're seeing this for the first time, even if you read about it last night, it's a mysterious thing, and one needs to see it from every possible point of view. It's something you get used to. You will never see it in a sudden flash of insight. It will just get as familiar to you as more common arithmetic, and algebraic, and calculus processes are.

But, look. What is it we demand? If you're going to call something an exponential, what is it we want an exponential to do, what gives an expression like this the right to be called e^(i theta)? The answer is I can't creep inside Euler's mind. It must have been a very big day of his life. He had a lot of big days, but when he realized that that was the thing to write down as the definition of e^(i theta). But, what is it one wants of an exponential? Well, the high school answer surely is you want it to satisfy the exponential law. Now, to my shock, I realize a lot of people don't know. In my analysis class, these are some math majors, or graduate engineers in various subjects, and if I say prove such and such using the exponential law, I'm sure to get at least half a dozen e-mails asking me, what's the exponential law?

Okay, the exponential law is a^x * a^y = a^(x + y): the law of exponents. That's the most important reason why, that's the single most important thing about exponents, are the way one uses them. And, this is the exponential function, called the exponential function because all this significant stuff is in the exponents. All right, so it should satisfy-- we want, first of all, the exponential law to be true.

But that's not all. That's a high school answer. An MIT answer would be, I mean, why is e to the x such a popular function? Well, of course, it does satisfy the exponential law, but for us, an even more reasonable thing. It's the function, which, when you differentiate it, you get the same thing you started with. And, it's apart from a constant factor, the only such function. Now, in terms of differential equations, it means that it's the solution that e to the, let's be a little generous, make it e^ax. No, better not to use x because complex numbers tend to be called x + iy. Let's use t as a more neutral variable, which is standing outside the fray, as it were. It satisfies the relationship that it's the solution, if you like, to the differential equation.

That's a fancy way of saying it. dy / dt = ay. Now, of course, that is not unique. We could make it unique by putting in an initial value. So, if I want to get this function and not a constant times it, I should make this an initial value problem and say that y(0) = 1. And now, I will get only the function, e to the at. So, in other words, that characterizes this function. It's the only function in the whole world that has that property.

Now, if you're going to call something e to the i theta, we want that to be true. So, here are my questions. Is it true that e to the i theta one, let's use that, times e^(i theta1) * e^(i theta2), see, I'm on a collision course here, but that's easily fixed. Is that equal to e^(i(theta1 + theta2))? If that turns out to be so, that's a big step. What would we like to be true here? Well, will it be true that d/dt (e^(i theta)) = i*e^(i theta).

So, question, question. I think those are the two most significant things. Now, the nodes do a third thing, talk about infinite series. Since we haven't done infinite series, anyway, it's not officially part of the syllabus, the kind of power series that are required. But, I will put it down for the sake of completeness, as people like to say. So, it should behave right. The infinite series should be nice. The infinite series should work out. There is no word for this, should work out, let's say. I mean, what's the little music? Is that some weird music idea, or is it only me that hears it? [LAUGHTER] Yes, Lord.

I feel I'm being watched up there. This is terrible. So, there's one guy. Here's another guy. And, I won't put a box around the infinite series, since I'm not going to say anything about it. Now, these things, in fact, are both true. Otherwise, why would I be saying them, and why would Euler have made the formula? But, what's interesting to see is what's behind them. And, that gives you little practice also in calculating with the complex numbers. So, let's look at the first one. What will it say? It is asking the question. It says, please, calculate the product of these two things. Okay, I do it, I'm told.

I will calculate cos(theta 1) + i sin(theta1). That's e^(i theta1), right? So, that corresponds to this. The other factor times the other factor, cos(theta2) + i sin(theta2). Okay, what does that come out to be? Well, again, we will use the method of grouping. What's the real part of it? The real part of it is cos(theta1)*cos(theta2). And then, there's a real part, which comes from these two factors. It's going to occur with a minus sign because of the i squared. And, what's left is sin(theta1)*sin(theta2). And then, the imaginary part, I'll factor out the i.

And then, what's left, I won't have to keep repeating the i. So, it will have to be sin(theta1)*cos(theta2). And, the other factor will be cosine theta one sine theta two-- + sin(theta2)*cos(theta1). Well, it looks like a mess, but, again, high school to the rescue. What is this? The top thing is nothing in disguise, but it's a disguised form of cos(theta1 + theta2). And the bottom is sin(theta1 + theta2). So, the product of these two things is this, and that's exactly the formula. In other words, this formula is a way of writing those two trigonometric identities for the cosine of the sum and the sine of the sum.

Instead of the two identities taking up that much space, written one after the other, they take up as much space, and they say exactly the same thing. Those two trigonometric identities are exactly the same as saying that e^(i theta) satisfies the exponential law. Now, people ask, you know, what's beautiful in mathematics? To me, that's beautiful. I think that's great. Something long turns into something short, and it's just as good, and moreover, connects with all these other things in the world, differential equations, infinite series, blah, blah, blah, blah, blah. Okay, I don't have to sell Euler. He sells himself. Now, how about the other one? How about the other one?

Now, that's obviously, I haven't said something because for one thing, how do you differentiate if there's theta here, and t down there. Okay, that's easily fixed. But, how do I differentiate this? What kind of a guy is e^(i theta)? Well, if I write it out, take a look at what it is. It's cos(theta) + i sin(theta). As theta varies, it's a function. The variable is real. Theta is a real variable. Its angle in radians, but it runs from negative infinity to infinity. So, if you think of functions as a black box, what's going in is a real number.

But, what's coming out is a complex number. So, schematically, here is the e^(i theta) box, if you like to think that way, theta goes in, and that's real, and a complex number, this particular complex number goes out. So, one, we'd call it, I'm not going to write this down because it's sort of pompous and takes too long. But, it is a complex valued function of a real variable.

You got that? Up to now, we studied real functions of real variables. But now, real valued functions of real variables, those are the kind calculus is concerned with. But now, it's a complex-valued function because the variable is real. But, the output, the value of the function is a complex number. Now, in general, such a function, well, maybe a better say, complex-valued, how about complex-valued function of a real variable, let's change the name of the variable.

t is always a real variable. I don't think we have complex time yet, although I'm sure there will be someday. But, the next Einstein appears. A complex-valued function of a real variable, t, in general, would look like this. t goes in, and what comes out? Well: a complex number, which I would then have to write this way. In other words, the real part depends on t, and the imaginary part depends upon t. So, a general function looks like this, a general complex-valued function. This is just a special case of it, where the variable has a different name. But, the first function would be cosine t, and the second function would be sine t. So, my only question is, how do you differentiate such a thing? Well, I'm not going to fuss over this.

The general definition is, with deltas and whatnot, but the end result of a perfectly fine definition is, you differentiate it by differentiating each component. The reason you don't have to work so very hard is because this is a real variable, and I already know what it means to differentiate a function of a real variable. So, I could write it this way, that the derivative of u plus iv, I'll abbreviate it that way, this means the derivative, with respect to whatever variable, since I didn't tell you what the variable in these functions were, well, I don't have to tell you what I'm differentiating with respect to. It's whatever was there because you can't see. And the answer is, it would be the derivative of u plus i times the derivative of v.

You differentiate it just the way you would if these were the components of a motion vector. You would get the velocity by differentiating each component separately. And, that's what you're doing here. Okay, now, the importance of that is that it at least tells me what it is I have to check when I check this formula. So, let's do it now that we know what this is. We know how to differentiate the function. Let's actually differentiate it.

That's fortunately, by far, the easiest part of the whole process. So, let's do it. So, what's the derivative? Let's go back to t, our generic variable. I want to emphasize that these functions, when we write them as functions, that theta will almost never be the variable outside of these notes on complex numbers. It will normally be time or something like that, or x, a neutral variable like x. So, what's the derivative of e^(i theta)? I'm hoping that it will turn out to be i e^(i theta), and that the yellow law may be true just as the green one was. Okay, let's calculate it. It's the derivative, with respect to, unfortunately I can convert t's to thetas, but not thetas to t's.

C'est la vie, okay. Times cos(t) + i sin(t), and what's that? Well, the derivative of cosine t, differentiating the real and imaginary parts separately, and adding them up. It's -sin(t) + i cos(t). Now, let's factor out at the i, because it says if I factor out the i, what do I get? Well, now, the real part of what's left would be cosine t. And, how about the imaginary part? Do you see, it will be i sin(t) because i*i = -1. And, what's that? e to the it. i * e^(i t). So, that works too.

What about the initial condition? No problem. What is y of zero? What's the function at zero? Well, don't say right away, i times zero is zero, so it must be one. That's illegal because, why is that illegal? It's because in that formula, you are not multiplying i times theta. I mean, sort of, you are, but that formula is the meaning of e^(i theta). Now, it would be very nice if this is like, well, anyway, you can't do that. So, you have to do it by saying it's the cos(0) + i sin(0). And, how much is that? The sine of zero is zero.

Now, it's okay to say i times zero is zero because that's the way complex numbers multiply. What is the cosine of zero? That's one. So, the answer, indeed, turns out to be one. So, this checks, really, from every conceivable standpoint down as I indicated, also from the standpoint of infinite series. So, we are definitely allowed to use this. Now, the more general exponential law is true. I'm not going to say much about it. So, in other words, e to the a, this is really a definition. e^(a + ib) is going to be, in order for the general exponential law to be true, this is really a definition. It's e^a * e^(ib). Now, notice when I look at the-- at any complex number, --

-- so, in terms of this, the polar form of a complex number, to draw the little picture again, if here is our complex number, and here is r, and here is the angle theta, so the nice way to write this complex number is r e^(i theta). The e^(i theta) is, now, why is that? What is the magnitude of this? This is r. The length of the absolute value, I didn't talk about magnitude in argument. I guess I should have. But, it's in the notes. So, r is called the modulus. Well, the fancy word is the modulus. And, we haven't given the complex number a name. Let's call it alpha, modulus of alpha, and theta is called, it's the angle.

It's called the argument. I didn't make up these words. There, from a tradition of English that has long since vanished, when I was a kid, and you wanted to know what a play was about, you looked in the playbill, and it said the argument of the play, it's that old-fashioned use of the word argument. Argument means the angle, and sometimes that's abbreviated by arg(alpha). And, this is abbreviated, of course, as absolute value of alpha, its length. Okay, the notes give you a little practice changing things to a polar form.

I think we will skip that in favor of doing a couple of other things because that's pretty easy. But let me, you should at least realize when you should look at polar form. The great advantage of polar form is, particularly once you've mastered the exponential law, the great advantage of polar form is it's good for multiplication. Now, of course, you know how to multiply complex numbers, even when they are in the Cartesian form. That's the first thing you learn in high school, how to multiply (a

+ bi)(c + di). But, as you will see, when push comes to shove, you will see this very clearly on Friday when we talk about trigonometric inputs to differential equations, --

-- that the changing to complex numbers makes all sorts of things easy to calculate, and the answers come out extremely clear, whereas if we had to do it any other way, it's a lot more work. And worst of all, when you finally slog through to the end, you fear you are none the wiser. It's good for multiplication because the product, so here's any number in its polar form. That's a general complex number. It's modulus, r1 e^(i theta) * r2 e^(i theta2). Well, you just multiply them as ordinary numbers. So, the part out front will be r1 r2, and the e^(i theta) parts gets multiplied by the exponential law and becomes e^(i (theta1 + theta2)) --

-- which makes very clear that the multiply geometrically two complex numbers, you multiply the moduli, the r's, the absolute values, how long the arrow is from zero to the complex number, multiply the moduli, and add the arguments. So the new number, its modulus is the product of r1*r2. And, its argument, its angle, polar angle, is the sum of the old two angles. And, you add the angles. And, you put down in your books angles, but I'm being photographed, so I'm going to write arguments.

In other words, it makes the geometric content of multiplication clear, in a sense in which this is extremely unclear. From this law, blah, blah, blah, blah, blah, whatever it turns out to be, you have not the slightest intuition that this is true about the complex numbers. That first thing is just a formula, whereas this thing is insightful representation of complex multiplication. Now, I'd like to use it for something, but before we do that, let me just indicate how just the exponential notation enables you to do things in calculus, formulas that are impossible to remember from calculus. It makes them very easy to derive. A typical example of that is, suppose you want to, for example, integrate e^(-x) cos(x).

Well, number one, you spend a few minutes running through a calculus textbook and try to find out the answer because you know you are not going to remember how to do it. Or, you run to a computer, and type in Matlab and something. Or, you fish out your little pocket calculator, which will give you a formula, and so on. So, you have aides for doing that. But, the way to do it if you're on a desert island, and the way I always do it because I never have any of these little aides around, and I cannot trust my memory, probably a certain number of you remember how you did it at high school, or how you did it in 18.01, if you took it here. You have to use integration by parts.

But, it's one of the tricky things that's not required on an exam because you had to use integration by parts twice in the same direction, and then suddenly by comparing the end product with the initial product and writing an equation. Somehow, the value falls out. Well, that's tricky. And it's not the sort of thing you can waste time stuffing into your head, unless you are going to be the integration bee during IAP or something like that.

Instead, using complex numbers is the way to do this. How do I think of this, cosine x? What I do, is I think of that e^(-x) cos(x) is the real part, the real part of what? Well, cosine x is the real part of e^(ix). So, this thing, this is real. This is real, too. But I'm thinking of it as the real part of e^(ix). Now, if I multiply these two together, this is going to turn out to be, therefore, the real part of e^(-x). I'll write it out very pompously, and then I will fix it. I would never write this, you are you.

Okay, it's e to the minus x times, when I write cosine x plus i sine x, so it is the real part of that is cosine x. So, it's the real part of, write it this way for real part of e to the, factor out the x, and what's up there is $(-1 + i)x$. Okay, and now, so, the idea is the same thing is going to be true for the integral. This is going to be the real part of that, the integral of $e^{(-1 + i)x}\,dx$. In other words, what you do is, this procedure is called complexifying the integral. Instead of looking at the original real problem, I'm going to turn it into a complex problem by turning this thing into a complex exponential.

This is the real part of that complex exponential. Now, what's the advantage of doing that? Simple. It's because nothing is easier to integrate than an exponential. And, though you may have some doubts as to whether the familiar laws work also with complex exponentials, I assure you they all do. It would be lovely to sit and prove them. On the other hand, I think after a while, you find it rather dull. So, I'm going to do the fun things, and assume that they are true because they are. So, what's the integral of $e^{(-1 + i)x}\,dx$? Well, if there is justice in heaven, it must be $e^{(-1 + i)x} / (-1 + i)$.

In some sense, that's the answer. This does, in fact, give that. That's correct. I want the real part of this. I want the real part because that's the way the original problem was stated. I want the real part only. So, I want the real part of this. Now, this is what separates the girls from the women. [LAUGHTER] This is why you have to know how to divide complex numbers. So, watch how I find the real part. I write it this way. Normally when I do the calculations for myself, I would skip a couple of these steps.

But this time, I will write everything out. You're going to have to do this a lot in this course, by the way, both over the course of the next few weeks, and especially towards the end of the term where we get into a complex systems, which involve complex numbers. There's a lot of this. So, now is the time to learn to do it, and to feel skillful at it. So, it's this times $e^{-x} * e^{ix}$, which is $\cos(x) + i \sin(x)$.

Now, I'm not ready, yet, to do the calculation to find the real part because I don't like the way this looks. I want to go back to the thing I did right at the very beginning of the hour, and turn it into an $a + bi$ type of complex number. In other words, what we have to do is the division. So, the division is going to be, now, I'm going to ask you to do it in your head. I multiply the top and bottom by negative one minus I. What does that put in the denominator? $1^2 + 1^2 = 2$. And in the numerator, $-1 - i$. This is the same as that. But now, it looks at the form $a + bi$. It's $-1 / 2 - i*(1/2)$.

So, this is multiplied by $e^{-x} \cos(x)$. So, if you are doing it, and practice a little bit, please don't put in all these steps. Go from here; well, I would go from here to here by myself. Maybe you shouldn't. Practice a little before you do that. And now, what do we do with this? Now, this is in a form to pick out the real part. We want the real part of this.

So, you don't have to write the whole thing out as a complex number. In other words, you don't have to do all the multiplications. You only have to find the real part of it, which is what? Well, e^{-x} will be simply a factor. That's a real factor, which I don't have to worry about. And, in that category, I can include the two also. So, I only have to pick out the real part of this times that. And, what's that?

It's -cos(x). And, the other real part comes from the product of these two things. i * -i = 1. And, what's left is sin(x). So, that's the answer to the question. That's integral e^(-x) cos x dx. Notice, it's a completely straightforward process. It doesn't involve any tricks, unless you call going to the complex domain a trick.

But, I don't. As soon as you see in this course the combination of e^(ax) cos(bx) or sin(bx), you should immediately think, and you're going to get plenty of it in the couple of weeks after the exam, you are going to get plenty of it, and you should immediately think of passing to the complex domain. That will be the standard way we solve such problems. So, you're going to get lots of practice doing this. But, this was the first time. Now, I guess in the time remaining, I'm not going to talk about in the notes, i, R, at all, but I would like to talk a little bit about the extraction of the complex roots, since you have a problem about that and because it's another beautiful application of this polar way of writing complex numbers.

Suppose I want to calculate. So, the basic problem is to calculate the nth roots of one. Now, in the real domain, of course, the answer is, sometimes there's only one of these, one itself, and sometimes there are two, depending on whether n is an even number or an odd number. But, in the complex domain, there are always n answers as complex numbers. One always has n nth roots. Now, where are they? Well, geometrically, it's easy to see where they are. Here's the unit circle. Here's the unit circle.

One of the roots is right here at one. Now, where are the others? Well, do you see that if I place, let's take n = 5 because that's a nice, dramatic number. If I place these peptides equally spaced points around the unit circle, so, in other words, this angle is alpha. Alpha should be the angle. What would be the expression for that? If there were five such equally spaced, it would be one fifth of all the way around the circle. All the way around the circle is two pi. So, it will be one fifth of two pi in radians. Now, it's geometrically clear that those are the five fifth roots because, how do I multiply complex numbers?

I multiply the moduli. Well, they all have moduli one. So, if I take this guy, let's call that complex number, oh, I hate to give you, they are always giving you Greek notation. All right, why not torture you? Zeta. At least you will learn how to make a zeta in this period, small zeta, so that's zeta. There's our fifth root of unity. It's the first one that occurs on the circle that isn't the trivial one, one. Now, do you see that, how would I calculate zeta to the fifth? Well, if I write zeta in polar notation, what would it be? The modulus would be one, and therefore it will be simply, r = 1 for it because its length is one. Its modulus is one.

What's up here? I times that angle, and that angle is 2 pi/5. So, there's just, geometrically I see where zeta is. And, if I translate that geometry into the e^(i theta) form for the formula, I see that it must be that number. Now, let's say somebody gives you that number and says, hey, is this the fifth root of one? I forbid you to draw any pictures.

What would you do? You say, well, I'll raise it to the fifth power. What's zeta to the fifth power? Well, it's e^(i 2 pi / 5), and now, if I think of raising that to the fifth power, by the exponential law, that's the same thing as putting a five in front of the exponent. So, this times five, and what's that? That's e^(i * 2pi). And, what is that?

Well, it's the angle. If the angle is two pi, I've gone all the way around the circle and come back here again. I've got the number one. So, this is one. Since the argument, two pi, is the same as an angle, it's the same as, well, let's not write it that way. It's wrong. It's just wrong since two pi and zero are the same angle. So, I could replace this by zero. Oh dear. Well, I guess I have to stop right in the middle of things. So, you're going to have to read a little bit about how to find roots in order to do that problem. And, we will go on from that point Friday.

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