Sinusoidally Driven Systems: Second Order LTI DE's

We start with the second order linear constant coefficient (CC) DE, which as we've seen can be interpreted as modeling a **damped forced harmonic oscillator**. If we further specify the oscillator to be a mechanical system with mass m, damping coefficient b, spring constant k, and with a *sinusoidal* driving force $B\cos\omega t$ (with B constant), then the DE is

$$mx'' + bx' + kx = B\cos\omega t. \tag{1}$$

For many applications it is of interest to be able to predict the periodic response of the system to various values of ω. From this point of view we can picture having a *knob* you can turn to set the input frequency ω, and a screen where we can see how the shape of the system response changes as we turn the ω-knob.

In the sessions on Exponential Response and Gain & Phase Lag we worked out the general case of a sinusoidally driven LTI DE. Specializing these results to the second order case we have:

Characteristic polynomial: $p(s) = ms^2 + bs + k$.

Complex replacement: $mz'' + bz' + kz = Be^{i\omega t}$, $\ x = \mathrm{Re}(z)$.

Exponential Response Formula:

$$z_p = \frac{Be^{i\omega t}}{p(i\omega)} = \frac{Be^{i\omega t}}{k - m\omega^2 + ib\omega}$$

$$\Rightarrow\ x_p = \mathrm{Re}(z_p) = \frac{B}{\sqrt{(k-m\omega^2)^2 + b^2\omega^2}}\cos(\omega t - \phi),$$

where $\phi = \mathrm{Arg}(p(i\omega)) = \tan^{-1}\left(\frac{b\omega}{k - m\omega^2}\right)$. (In this case ϕ must be between 0 and π. We say ϕ is in the first or second quadrants.)

Letting $A = \frac{B}{\sqrt{(k-m\omega^2)^2 + b^2\omega^2}}$, we can write the periodic response x_p as

$$x_p = A\cos(\omega t - \phi).$$

The *complex gain*, which is defined as the ratio of the amplitude of the output to the amplitude of the input in the *complexified* equation, is

$$\tilde{g}(\omega) = \frac{1}{p(i\omega)} = \frac{1}{k - m\omega^2 + ib\omega}.$$

The *gain*, which is defined as the ratio of the amplitude of the output to the amplitude of the input in the *real* equation, is

$$g = g(\omega) = \frac{1}{|\ p(i\omega)\ |} = \frac{1}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}. \tag{2}$$

The *phase lag* is

$$\phi = \phi(\omega) = \operatorname{Arg}(p(i\omega) = \tan^{-1}(\frac{b\omega}{k - m\omega^2}) \tag{3}$$

and we also have the *time lag* = ϕ/ω.

Terminology of Frequency Response
We call the gain $g(\omega)$ the **amplitude response** of the system. The phase lag $\phi(\omega)$ is called the **phase response** of the system. We refer to them collectively as the **frequency response** of the system.

Notes:
1. Observe that the whole DE scales by the input amplitude B.

2. All that is needed about the input for these formulas to be valid is that it is of the form (*constant*) $\times$ (a *sinusoidal* function). Here we have used the notation $B\cos\omega t$ but the amplitude factor in front of the cosine function can take any form, including having the constants depend on the system parameters and/or on ω. (And of course one could equally-well use $\sin\omega t$, or any other shift of cosine, for the sinusoid.) This point is very important in the physical applications of this DE and we will return to it again in a later session.

3. Along the same lines as the preceding: we always define the gain as the *the amplitude of the periodic output divided by the amplitude of the periodic input*. Later in this session we will see examples where the gain is *not* just equal to $\frac{1}{p(i\omega)}$ (for complex gain) or $\frac{1}{|p(i\omega)|}$ (for real gain) – stay tuned!