First order Unit Step Response

1. Unit Step Response

Consider the initial value problem

$$\dot{x} + kx = ru(t), \quad x(0^-) = 0, \quad k, r \text{ constants.}$$

This would model, for example, the amount of uranium in a nuclear reactor where we add uranium at the constant rate of r kg/year starting at time $t = 0$ and where k is the decay rate of the uranium.

As in the previous note, adding an infinitesimal amount ($r\,dt$) at a time leads to a continuous response. We have $x(t) = 0$ for $t < 0$; and for $t > 0$ we must solve

$$\dot{x} + kx = r, \; x(0) = 0.$$

The general solution is $x(t) = (r/k) + ce^{-kt}$. To find c, we use $x(0) = 0$:

$$0 = x(0) = \frac{r}{k} + c \;\Rightarrow\; c = -\frac{r}{k}.$$

Thus, in both cases and u-format

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{r}{k}(1 - e^{-kt}) & \text{for } t > 0 \end{cases} \quad = \quad \frac{r}{k}(1 - e^{-kt})u(t). \tag{1}$$

With $r = 1$, this is the **unit step response**, sometimes written $v(t)$. To be more precise, we could write $v(t) = u(t)(1/k)(1 - e^{-kt})$.

The claim that we get a continuous response is true, but may feel a bit unjustified. Let's redo the above example very carefully without making this assumption. Naturally, we will get the same answer.

The equation is

$$\dot{x} + kx = \begin{cases} 0 & \text{for } t < 0 \\ r & \text{for } t > 0, \end{cases} \qquad x(0^-) = 0. \tag{2}$$

Solving the two pieces we get

$$x(t) = \begin{cases} c_1 e^{-kt} & \text{for } t < 0 \\ \frac{r}{k} + c_2 e^{-kt} & \text{for } t > 0. \end{cases}$$

This gives $x(0^-) = c_1$ and $x(0^+) = r/k + c_2$. If these two are different there is a jump at $t = 0$ of magnitude

$$x(0^+) - x(0^-) = r/k + c_2 - c_1.$$

The initial condition $x(0^-) = 0$ implies $c_1 = 0$, so our solution looks like

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{r}{k} + c_2 e^{-kt} & \text{for } t > 0. \end{cases}$$

To find c_2 we substitute this into our differential equation (2). (We must use the generalized derivative if there is a jump at $t = 0$.) After substitution the left side of (2) becomes

$$\begin{aligned} \dot{x} + kx &= (r/k + c_2)\delta(t) + \begin{cases} 0 & \text{for } t < 0 \\ -kc_2e^{-kt} + r + kc_2e^{-kt} & \text{for } t > 0 \end{cases} \\ &= (r/k + c_2)\delta(t) + \begin{cases} 0 & \text{for } t < 0 \\ r & \text{for } t > 0. \end{cases} \end{aligned}$$

Comparing this with the right side of (2) we see that $r/k + c_2 = 0$, or $c_2 = -r/k$. This gives exactly the same solution (1) we had before.

Figure 1 shows the graph of the unit step response ($r = 1$). Notice that it starts at 0 and goes asymptotically up to $1/k$.

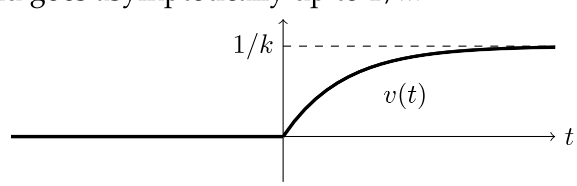

Figure 1. Unit step is the response of the system $\dot{x} + kx = f(t)$ when $f(t) = u(t)$.

The Meaning of the Phrase 'Unit Step Response'
In this note looked at the system with equation

$$\dot{x} + kx = f(t)$$

and we considered $f(t)$ to be the input. As we have noted previously, it sometimes makes more sense to consider something else to be the input. For example, in Newton's law of cooling

$$\dot{T} + kT = kT_e$$

it makes physical sense to call T_e, the temperature of the environment, the input. In this case the unit step response of the system means the response to the *input* $T_e(t) = u(t)$, i.e. the solution to

$$\dot{T} + kT = ku(t).$$