18.03SC Differential Equations, Fall 2011

Transcript – Linear Systems: Complex Roots

PROFESSOR: Welcome to this recitation. In this recitation, we're going to look at linear systems with complex roots. So the system we're examining is the one giving us x dot equals minus 3x minus 2, and y dot equals 5x minus y. And you're asked to use the matrix methods to solve this system. So why don't you take a pause here and try to solve this problem? And I'll be right back.

Welcome back. So the first step is to write this system in matrix form. So we introduced a vector xy, the matrix multiplying column vector xy again. The coefficients are going to be minus 3, minus 2, 5, minus 1. So the first step in solving this system is to find the eigenvalues of the matrix A.

So the eigenvalues of matrix A are basically the solutions to this following determinant equal to 0, minus 3 minus lambda minus 2, 5, minus 1 minus lambda, determinant equals to 0. Here, the lambda are the unknown eigenvalues. And to get this determinant, we're basically multiplying these two terms, minus minus 2 dot 5, which gives us a plus sign. So here, we're going to have lambda squared. 3 lambda plus 1 lambda gives us 4 lambda. And 3 plus 10 gives us 13.

So this second order polynomial in lambda will give us the two eigenvalues for this matrix. So let's examine the discriminant. So we have v squared minus 4ab. And this gives us minus 36. So the discriminant is negative. And that tells us that we are going to have two complex roots, which is the title of the recitation. And these two complex roots are going to be complex conjugate of each other.

So formula gives us plus or minus i of 6 for the root of the discriminant. Here, we have minus 4 over 2. So the two roots are basically minus 2 plus or minus i3 or 3i. So these are our two roots. So now, let's focus on one of the roots to get the eigenvector associated with the eigenvalue. So let's focus on the positive one, for example. And we could do all the following again--

AUDIENCE: Minus 2.

PROFESSOR: Minus 2. Thank you. We can do all the following calculation that I'm going to go do now for the complex conjugate, and I will explain at the end how that basically not change the result. So for this eigenvalue, we need to compute now the eigenvector. So to do that, we basically have to use minus 3 minus-- I'm just going to write the system here, let you see what I'm doing. And we are solving this system.

So where does this system come from? It comes from the fact that we're looking for an eigenvector, v plus, that is defined as av plus equals lambda plus v plus. And you can then bring everything on the left-hand side, a minus lambda i applied to v plus, gives us the zero vector. So that's what we have here. The unknowns on a1, a2, and we're going to try to solve for this.

So if we plug in now for the value of lambda plus that we have, we have minus 3 plus 2, which gives us minus 1. And then, we have a minus 3i and minus 2. And for the second line, second entry of this matrix you have 5 minus 1 minus minus 2. So we have 2 minus 1, which is 1. And then, we have minus 3i. A1, a2 equals to 0, 0.

So here, you can check for yourself that these two equations given by the first line and the second line are actually the same. And so basically, to get a1 and a2, it is sufficient to just solve, for example, the first one, where here, I just wrote minus 1 minus 3i multiplied by a1. Minus 2 a2 equals to 0. And I just brought the minus 2 on this side. So here, you can see that if we take a1 equals to 0 equals to plus 2, which would be our first entry, we can then cancel out these two and just have a2 equals to minus 1 minus 3i. So this would be one eigenvector associated with this eigenvalue. We could have picked other ones. They're basically a parallel to this one.

So now what? So what we need to remember is the meaning of all of this. Seeking the eigenvalues and the eigenvectors is basically equivalent to seeking a solution in the form lambda t exponential lambda t with the direction of the eigenvector associated with this eigenvalue. So now that we actually have this eigenvector and this eigenvalue, we can write down the solution.

And I'm just going to write the solution in x, which has entries basically x and y. And one way of writing it would be just to basically first start by writing what we have there. I'm just going to spell it out. So we have this multiplied by 2 minus 3i. So what do we do with this? Well, we remember our earlier formula. So this is exponential minus 2t plus exponential 3it. So we can split the exponential 3it into a cosine and a sine. And this, we're going to also be able to split it into the complex part and the real part. And then, we're going to combine the real part and the complex part. So let's do that.

Exponential minus 2t multiplying, basically, cosine 3t plus i sine 3t for the entry 2 minus 1 minus 3i. So we have an i here and an i here. So things can be combined into a real part. So in the first entry here, what are we going to have? We're going to have cos 3t multiplying 2. That's going to be in the real part. And I'm going to keep some space. And another entry here at the second entry of this vector is going to give us cosine 3t multiplied by minus 1. Oops, here, it should be a 3t. Sorry. So minus cosine 3t.

Now, where are we going to have another real part here? It's going to come from a multiplication of i sine 3t by 3i. So the two i's together gives a minus 1. And we end up with a plus 3 sine 3t. So we're done for the real part. Now let's focus on the imaginary part. What do we have? We have an i sine 3t multiplying a 2. And we have a minus 3i here multiplying cosine 3t. So we want to have a minus 3 cosine 3t, and finally, this minus 1 multiplying this sine 3t. So now, we did--

AUDIENCE: [INAUDIBLE].

PROFESSOR: Oh yeah. Thank you. 2, from this operation. So now, we did split our solution into a real part and an imaginary part. So how can we write the general solution of the system? Well, we knew that for this linear system of equations. If we have a complex number that is a solution to be a linear equation, then its real part and its imaginary part are also two independent solutions. So we can write the general solution of the system as a linear combination of the real part and the imaginary part.

And I can just label this u1 of t and u2 of t here and vector. And we can then write the general solution terms of any constant that would be determined by the initial condition if we had one. Exponential minus 2t along u1 plus c2 exponential minus 2t along vector u2, which are also functions of t, just the difference from what we had before.

So here, basically we seek for the eigenvalues values of the matrix. We looked for the eigenvector associated with the complex eigenvalue. We were able to write the full solution. And then, because of the linearity property, we were able to just then extract two linearly independent solutions, the real part of the solution we had and the imaginary part of the solution we had.

So what I mentioned earlier was that we could do this whole calculation for the other eigenvalue with a minus. If you try to do it and trickle down your minus, you would see that basically you would just end up with minus signs here for basically in front of the sines. And what you could do then is just simply absorb that minus sign for the general solution in c_1 and c_2. And basically, it gives you exactly the same form for the general solution. So you don't need to redo it for the second one. You would still end up with only two linearly independent solutions, not four. OK, so that ends this recitation.

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