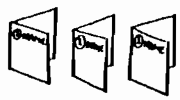
18.04 Ancient History #3

Mon 01 Dec 03

Just once more, please ...

18.04 Exam #3

CLOSED BOOK



Friday, December 4, 1987

① Show that the function

$$w(z) = z + \frac{1}{z}$$

maps any circle $|z|$ = const. from the z-plane into an ellipse in the w-plane, with foci at $w = \pm 2$.

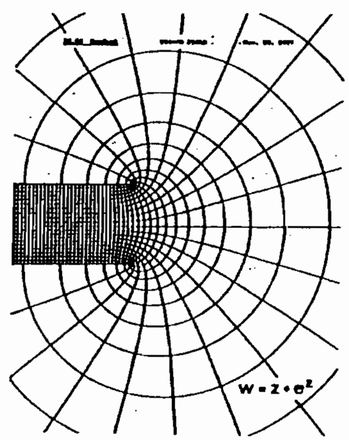
② a) Obtain a Fourier cosine series that approximates the function $f(t) = \sin^4 t$ to very high accuracy.

b) Use that series, along with some sort of a Fourier expansion also for $y(t)$, to figure out the periodic or steady-state solution of the differential equation

$$\frac{dy}{dt} + 3y = \sin^4 t \ .$$

③ As alleged in lecture, the transformation $w = z + e^z$ is of considerable value in studying the equipotentials and field lines which flare out near the edge of a parallel-plate capacitor more or less as pictured in this diagram:

Your task today: Determine the locations in midplane of this geometry at which the vertical electric field has values equal to exactly 90% and 10% of its asymptotic value deep between the plates to the far left. Cite your answers as multiples of the plate separation L .

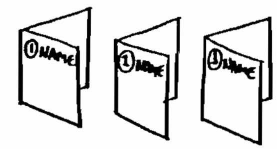


18.04 Exam #4

Monday, December 8, 1986

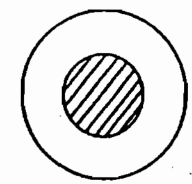
CLOSED BOOK

Just once more, please ...



① Solve $\nabla^2 T = 0$ within the annulus bounded by the concentric circles $|z| = 1$ and $|z| = 2$. Let $T = 0$ at all points on the inner circle, and $T = 5 \cos 3\theta$ on the outer circle.

HINTS: Think of z^n and z^{-n} .



② Figure out (for the munificent rewards of 4, 3, 2 and 1 pts., resp.) the coefficients a_o , a_2 , a_4 and a_6 needed in the series

$$|\sin x| = a_o + a_2 \cos 2x + a_4 \cos 4x + a_6 \cos 6x + \ldots$$

③ Recall from lecture that

$$x = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right] \quad \text{for } |x| < \pi .$$

a) Integrate both sides of this expression to obtain a similar Fourier series for the function $x^2 + C$, valid in the same interval.

b) Evaluate any integration constant above, by insisting that your new parabolic function and its Fourier imitation agree even as to mean values. (HINT: make a sketch!)

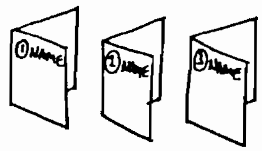
c) From the new formula applied at $x = 0$, evaluate the famous sum

$$1 - 1/4 + 1/9 - 1/16 + 1/25 - \ldots$$

18.04 Exam #4 Friday, December 6, 1985

CLOSED BOOK

As before, please ...



(1) Show explicitly that the function $w = z^2$ maps the straight line $x + y = 2$ from the z-plane into a certain parabola in the w-plane, and also that it maps both branches of the hyperbola $x^2 = 1 + y^2$ into a certain straight line. Of course, do tell which parabola, and which straight line.

(2) Determine the coefficients $a_0, a_1, a_2, \ldots$ needed to represent the function

$$f(x) = e^{-|x|} = \sum_{k=1}^{\infty} a_k \cos k\pi x$$

inside the interval $-1 < x < 1$ by means of the indicated Fourier cosine series of period 2 .

(3) (a) Presumably from some complex exponential, find that unique harmonic function $T(x,y)$ which tends to zero as $y \to +\infty$, and yet for which $T(x,y=0) = \cos nx$ at every x on the lower boundary $y = 0$.

(b) Now replace the boundary "temperature" of such a semi-infinite hot plate by the square wave

$$T(x,0) = \text{sqw}(x) = \frac{4}{\pi}\{\cos x - \tfrac{1}{3}\cos 3x + \tfrac{1}{5}\cos 5x - \ldots\}.$$

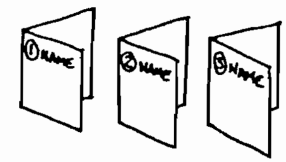
Use your answers from part (a) to determine an infinite series describing $T(x=0,y\geq 0)$ — i.e., the decreasing temperature profile simply along the y-axis .

BONUS: A rich reward of 3 extra points awaits anyone who correctly sums $T(0,y)$ from 3.b into a tidy function.

18.04 Exam #3 Friday, November 30, 1984

CLOSED BOOK

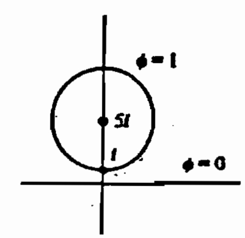
As in the past, please record your struggles with these three problems on separate sheets ...



(1) A rather lazy frog who is obviously "right-handed" leaps one meter eastward on his first jump, 1/2 meter on his second, 1/4 meter on his third, 1/8 meter on the fourth, and so forth, each time turning exactly an angle α to the left from his previous flight path. Assuming only that $0 < \alpha < \pi$, show that this fine fellow comes to rest invariably at some spot on a semicircle of radius 2/3 .

(2) Use residue calculus to evaluate again $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 - 4\cos\theta}$

(3) Probably via some bilinear transformation, find a function $\phi(x,y)$ that is harmonic in the portion of the upper half-plane exterior to the circle $|z - 5i| = 4$ and that has value $+1$ on that circle and value 0 on the real axis — just as shown in this figure copied from our textbook.



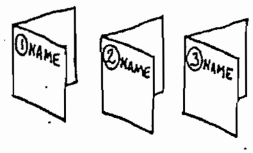
18.04 Modern History #3

Mon 01 Dec 03

18.04 Exam #4 — Friday, December 6, 2002

CLOSED BOOK ... and **NO calculators**

Once more, please ...



(1) Show us again, working pretty much from first principles, that both of the loci

(a) $\text{Re}\{ \log(z-1) - \log(z+1) \} = \ln 3$

and

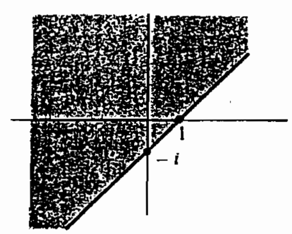
(b) $\text{Im}\{ \log(z-1) - \log(z+1) \} = \pi/2$

are either **circles** or arcs thereof. And just which x,y circles or arcs are they exactly?

(2) Find a **Möbius** transformation of type

$$w(z) = \frac{az + b}{cz + d}$$

that takes the shaded region $x - y < 1$ from the z-plane on the right into the interior of the unit disk $|w| < 1$.



(3) Find the typical coefficient c_n needed for the complex **Fourier series**

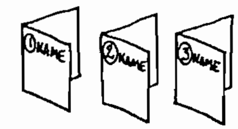
$$e^t = \ldots + c_{-1}e^{-it} + c_0 + c_1e^{it} + c_2e^{2it} + c_3e^{3it} + \ldots$$

to be valid over the interval $[-\pi,\pi]$. In particular, what will be the amplitudes a_2 and b_2 of the terms $a_2 \cos(2t)$ and $b_2 \sin(2t)$ in the corresponding real Fourier series, now expressed simply as **multiples** of the above mean value c_0 ?

18.04 Exam #3 — Friday, December 1, 2000

CLOSED BOOK ... and **NO calculators**

Once again, please ...



(1) Evaluate $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} \cos kx \, dx$,

given the HINT [ ] and FACT $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

(2) Show explicitly that the function $w = z^2$ maps the straight line $x + y = 2$ from the z-plane into a certain **parabola** in the w-plane , and also that it maps **both** branches of the hyperbola $x^2 = 1 + y^2$ into a certain **straight line**.

Of course specify which parabola, and which straight line.

(3) Use what by now you (ought to!) have learned about complex geometric series and inversion maps to polish off this old chestnut really neatly and convincingly:

> A rather lazy frog leaps one yard eastward on its first jump, 1/2 yard on its second, 1/4 yard on its third, 1/8 yard on the fourth, and so forth, each time turning exactly an angle α to the left from the preceding flight segment. Assuming only that $0 < \alpha < \pi$, show that this fine creature comes to rest invariably at some spot of a **semicircle of 2-foot radius** which we also ask you to mark clearly for everyone's benefit.

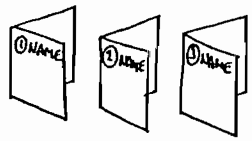
Invaluable HINT to metric scholars: 1 yard = 3 feet .

18.04 Exam #3

Friday, April 30, 1999

CLOSED BOOK ... and NO calculators

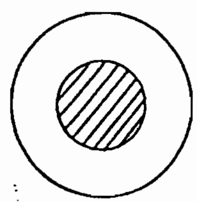
As before, please struggle with Problems 1, 2 and 3 on separate sheets of paper ...



(1) For $-1 < a < 1$, use residue theory to evaluate

$$I(a) = \int_0^\infty \frac{x^a}{x^2+1}\,dx .$$

(2) Solve $\nabla^2 T = 0$ within the annulus bounded by the concentric circles $|z| = 1$ and $|z| = 2$. Let $T = 0$ at all points on the inner circle, and $T = 5\cos 3\theta$ on the outer circle.



HINTS: Think of z^n and z^{-n}.

(3) Use our friend e^{ix} to evaluate neatly and efficiently:

(a) the integral $\int_0^{2\pi} \cos^8 x \, dx$

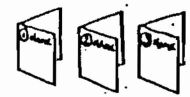
(b) the sum $\sum_{n=0}^{\infty} 3^{-n} \cos(nx)$

18.04 Exam #4

Friday, May 8, 1996

CLOSED BOOK

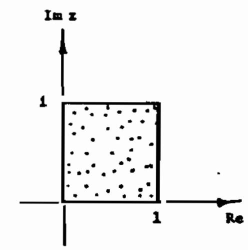
As before, please struggle with these problems on separate sheets of paper ...



(1) Kindly map this unit square from the complex z-plane onto the w-plane via the bilinear (= Möbius) transformation

$$w = \frac{z - i}{z + i} ,$$

and both sketch and carefully describe the resulting "squashed quadrilateral" which consists, as we know, of just the arcs of four circles.



(2) Combine your knowledge of Fourier series and of powers of z to find a real function $T(x,y)$ that is harmonic within the unit circle $x^2+y^2 = 1$ and that assumes the values $T = \cos^3\theta$ along the edge $x = \cos\theta$, $y = \sin\theta$.

Also report the specific values $T(\frac{1}{2},0)$ and $T(\frac{1}{2},\frac{1}{2})$.

(3) Determine the (surprisingly pleasant!) coefficients c_n needed in the Fourier expansion

$$\frac{4}{5 - 3\cos\theta} = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} .$$

HINTS: Don't even contemplate any hairy integrals here, much too tedious to do this job in the available few minutes, even using well-oiled residue calculus.

Instead, think of partial fractions, geometric series, etc.