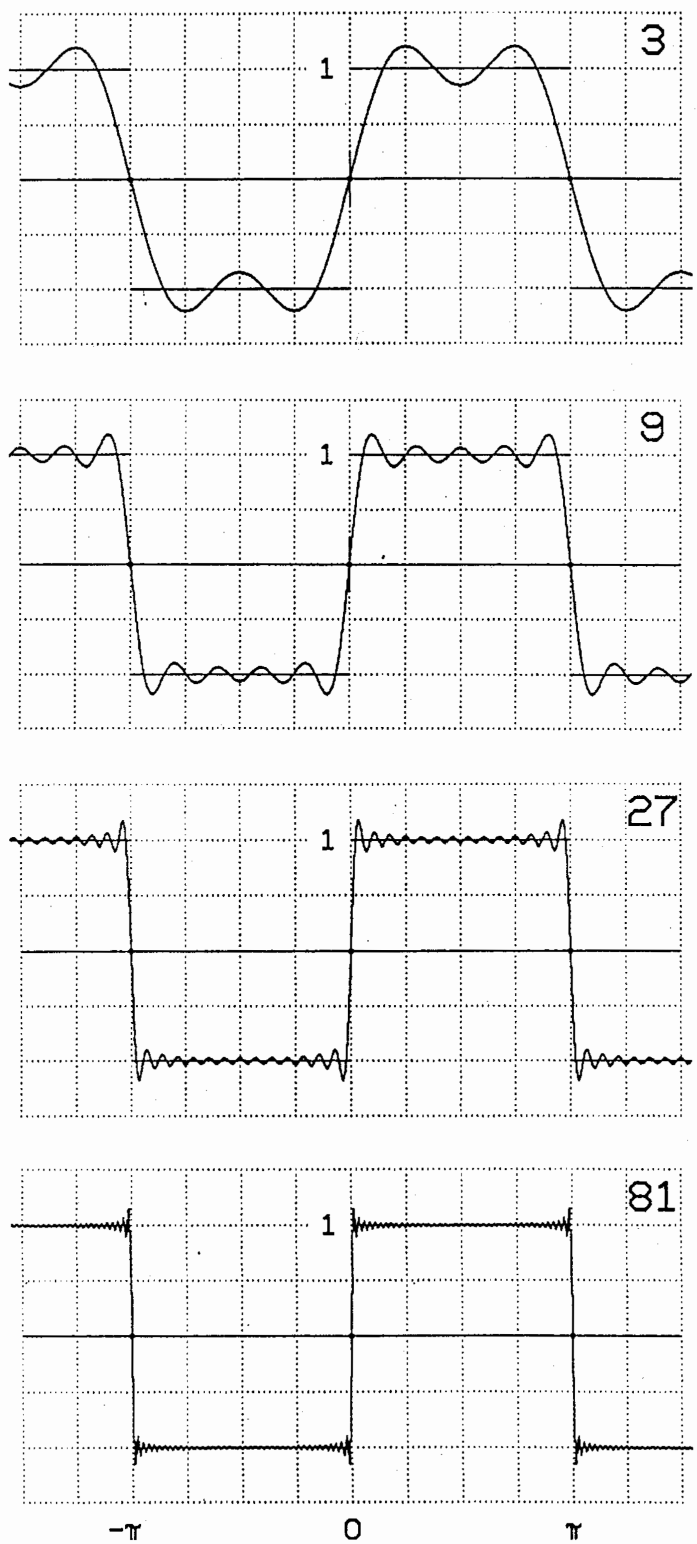
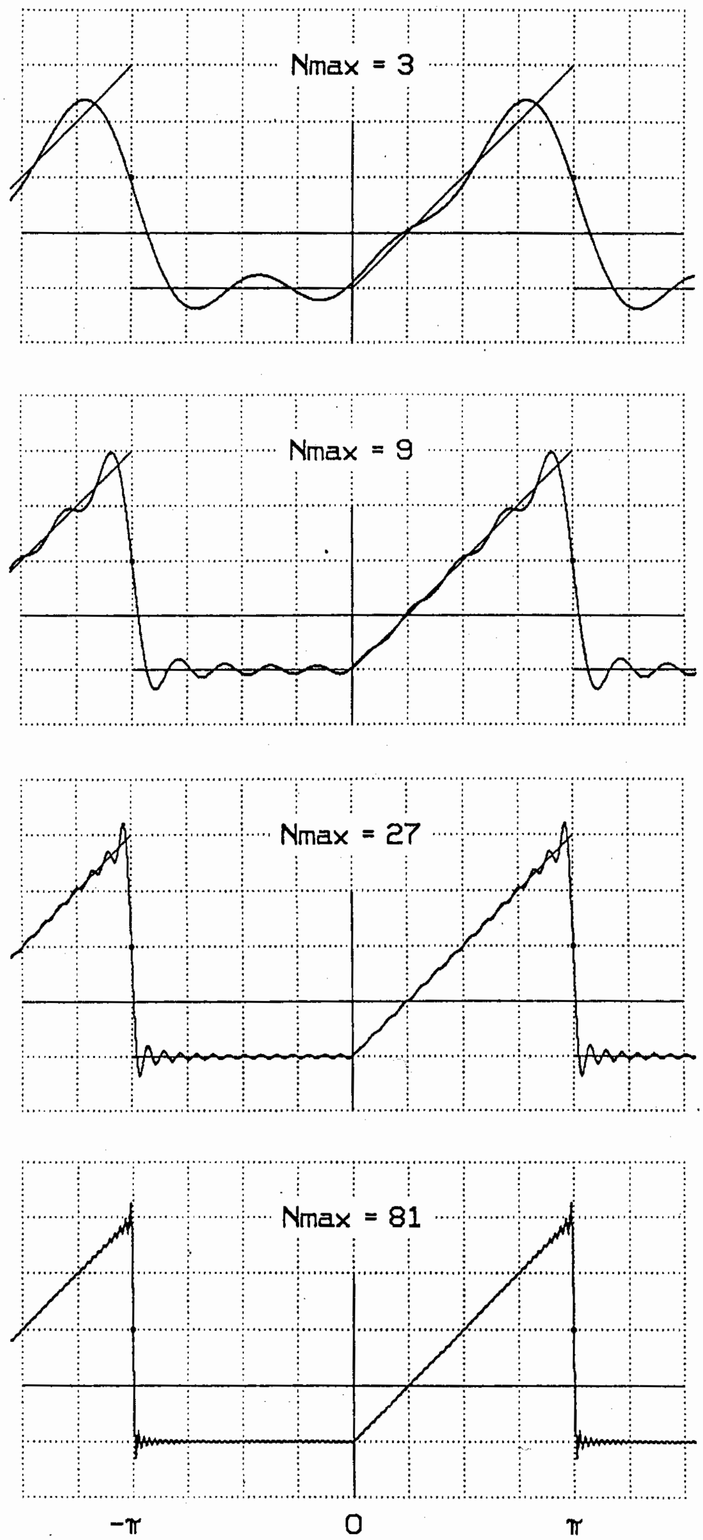
reused as 18.04 Handout Fri 21 Nov 03

18.03 Handout **FOURIER I** Wed 24 Oct 01

$$f(t) \sim \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin nt}{n} = \frac{4}{\pi} \left(\sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t + \cdots \right)$$

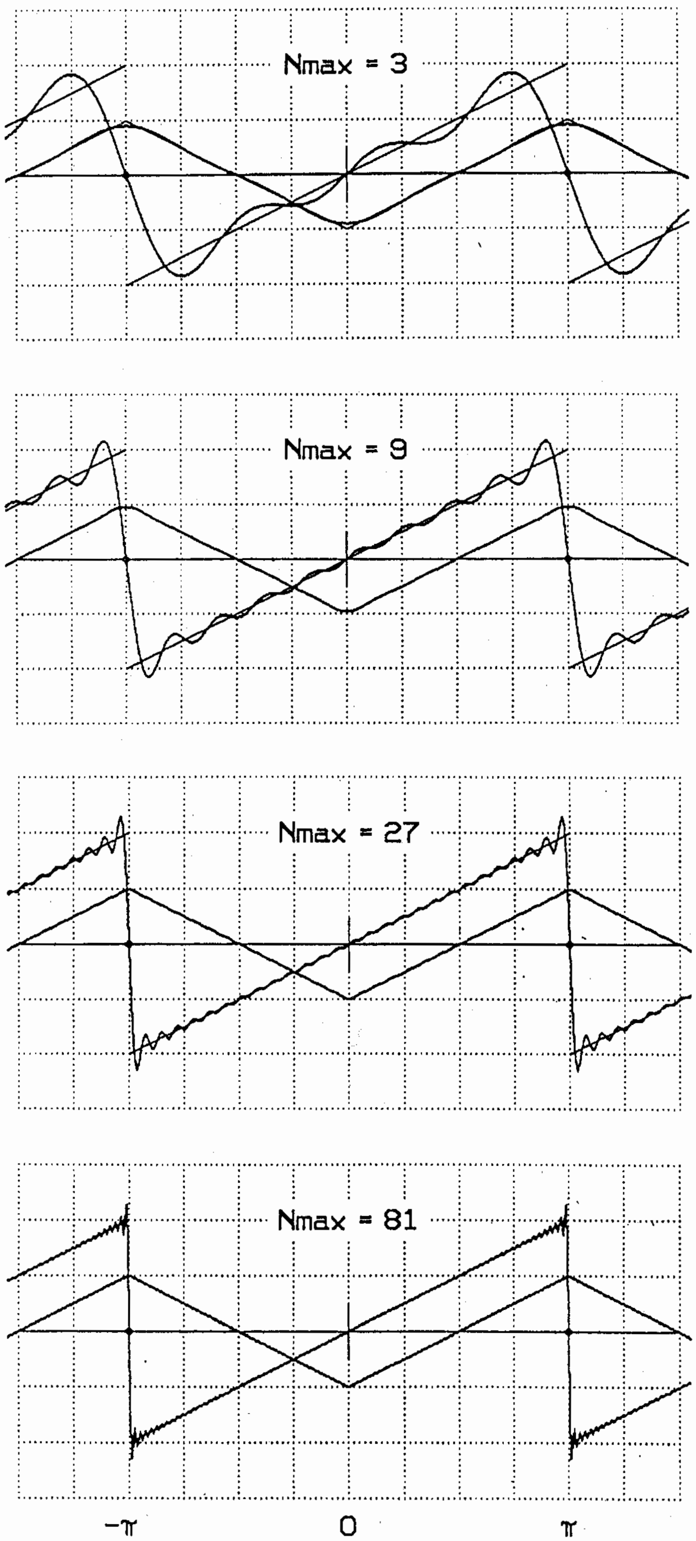


$$f(t) \sim \frac{\pi}{4} - \frac{2}{\pi}\sum_{n\ \mathrm{odd}} \frac{\cos nt}{n^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin nt}{n}$$



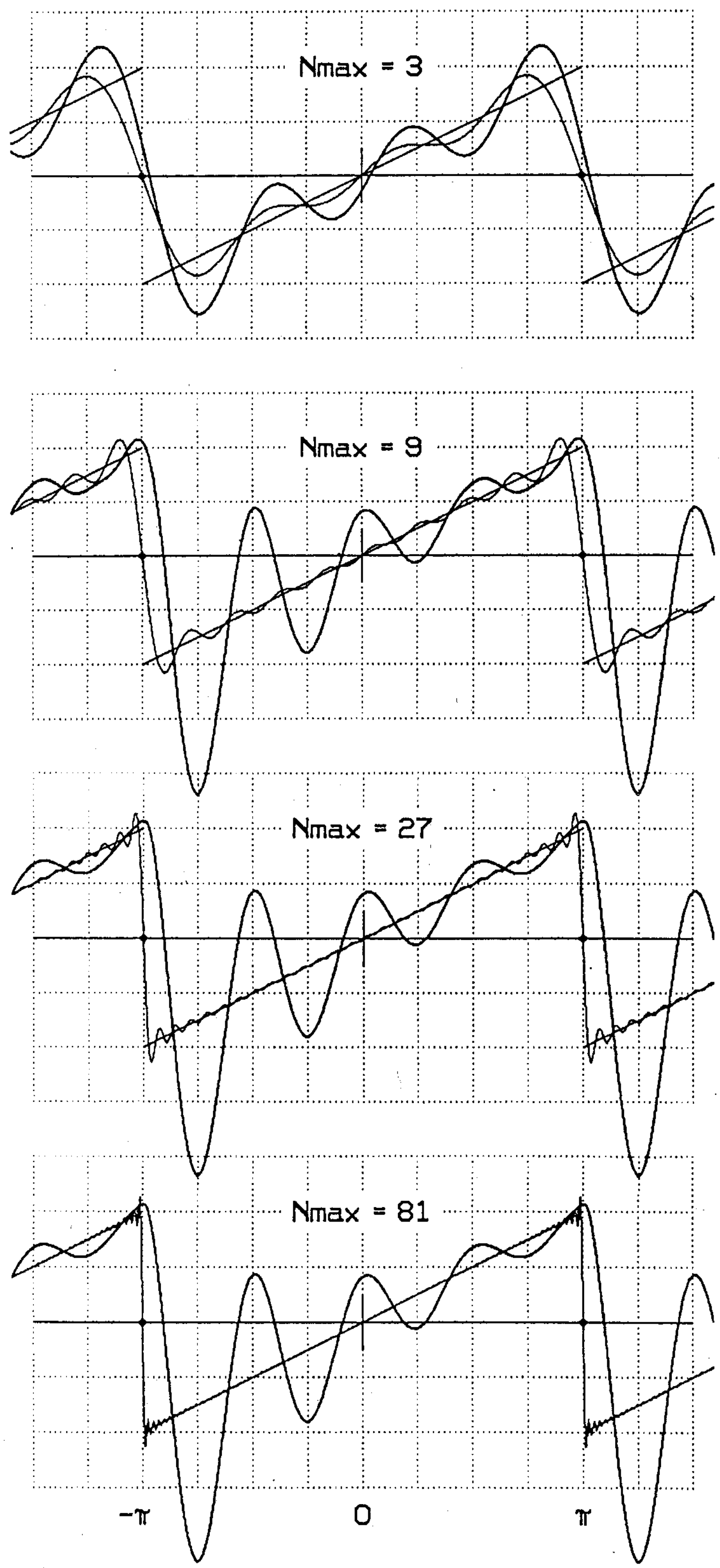
18.03 Handout FOURIER III Fri 26 Oct 01

Truncated FS for $-\dfrac{2}{\pi}\sum_{n\ \text{odd}}^{\text{Nmax}} \dfrac{\cos nt}{n^2}$ and $\sum_{n=1}^{\text{Nmax}} \dfrac{(-1)^{n+1}\sin nt}{n}$, plotted separately

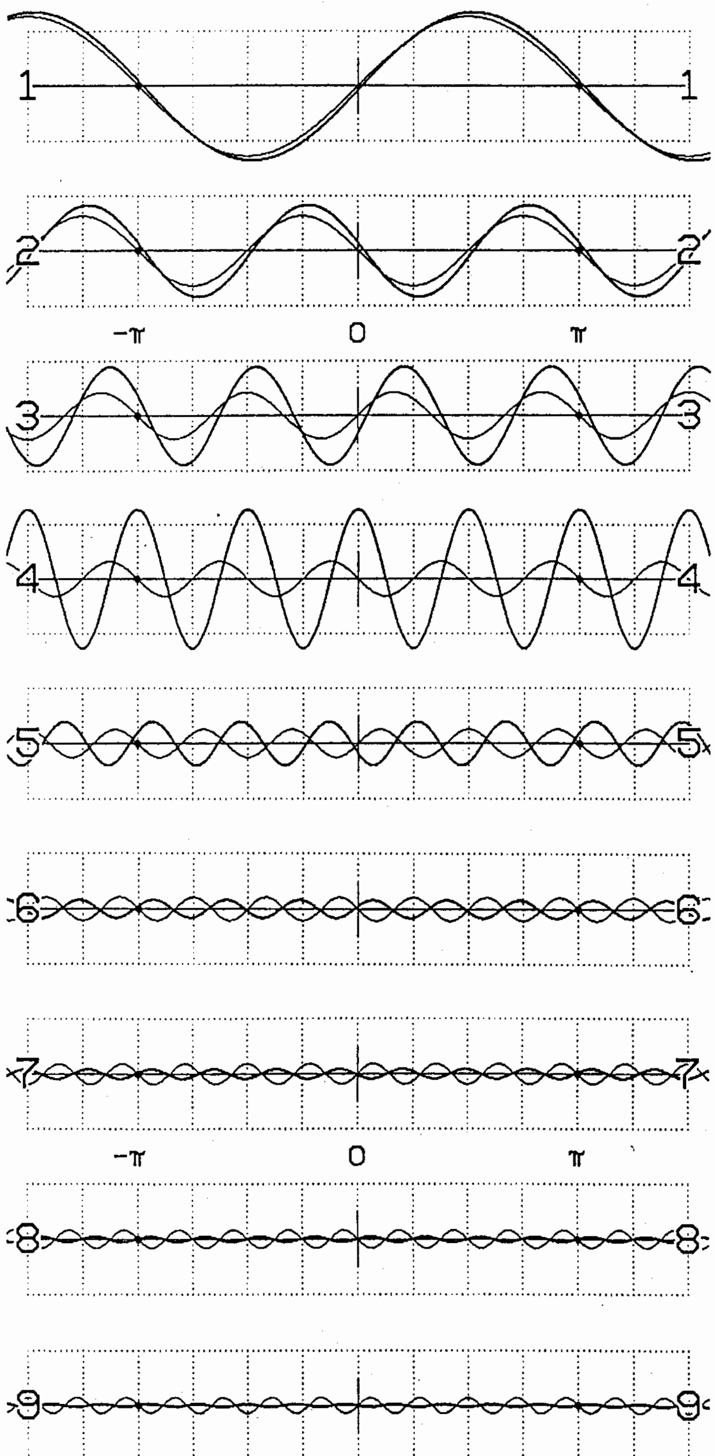


18.03 Handout **FOURIER - IV** Fri 26 Oct 01

Periodic Solns of $\ddot{y} + \dot{y} + 16y = 16 \sum_{n=1}^{\mathrm{Nmax}} \frac{(-1)^{n+1} \sin nt}{n}$



Harmonics for $\ddot{y} + \dot{y} + 16y = 16\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nt}{n}$



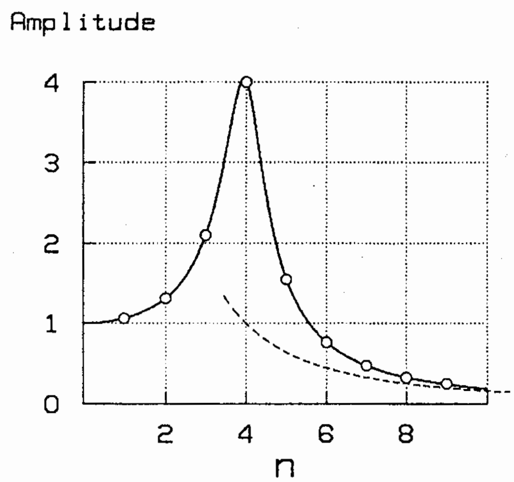
To appreciate the nine pairs of sinusoidal curves from on the front side, recall that the **periodic response** of the damped harmonic oscillator

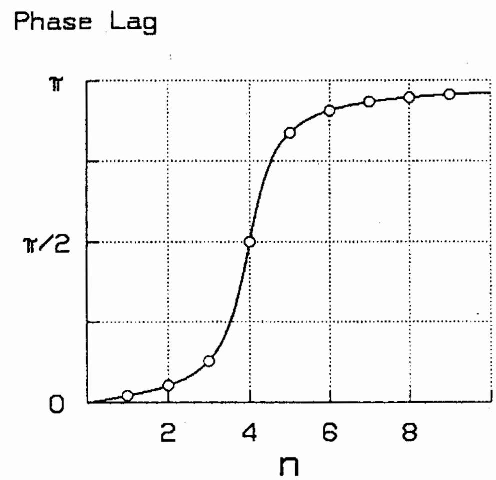
$$\ddot{y} + \dot{y} + 16y = 16f(t)$$

to forcing by any single sine wave $f(t) = \sin nt$ works out neatly as

$$y(t) = \text{Im}\left\{\frac{16e^{int}}{(16-n^2)+in}\right\} = P\sin nt - Q\cos nt \quad ,$$

where $P = 16(16-n^2)/\Delta$ and $Q = 16n/\Delta$, with denominator $\Delta = (16-n^2)^2 + n^2$. Equivalently, we expect this response $y(t) = A\sin(nt - \varphi)$ to have an **amplitude** $A(n) = \sqrt{P^2+Q^2}$ and **phase lag** $\varphi(n) = \pi/2 - \tan^{-1}(P/Q)$, both depending on the imposed frequency n much as pictured in these two small plots:





Thus exactly at **resonance** with $n = 4$, these formulas imply an amplitude $A = 4$ and phase lag $\varphi = 90°$. Similarly for $n = 3$ we expect $A = \sqrt{3712/841} = 2.101$ and $\varphi = \tan^{-1}(3/7) = 23.20°$, and for $n = 5$ we expect $A = \sqrt{6784/2809} = 1.554$ and $\varphi = \pi/2 + \tan^{-1}(9/5) = 150.95°$. For tiny values of n, moreover, the predicted amplitude approaches unity and the phase lag vanishes, whereas for large values of n the amplitude $A \sim 16/n^2$ and the phase lag $\varphi \to \pi$.

All this was neatly summarized for the separate harmonics $n = 1, 2, ..., 9$ by the curve pairs in the front, there including also the factor $(-1)^{n+1}/n$ that represents the coefficients of the Fourier sine series for our postulated forcing, namely the odd function $t/2$ of period 2π . Those single-n forcing harmonics were all plotted as the faint sinusoids passing through (0,0), and the inferred responses as the thicker curves.

Remarkably, just these first nine forced oscillations — each not changing at all in amplitude with time t — added up to yield the strongly **damped** overall response pictured in the `Nmax = 9` frame of our earlier handout titled `FOURIER IV` !