18.04 Handout re: FFT Wed 10 Dec 03

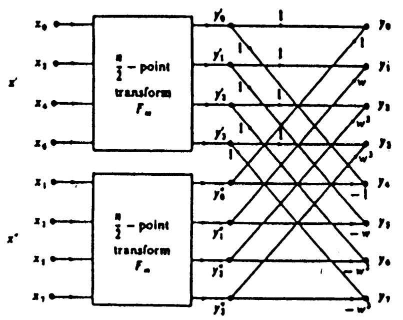
On the left below are some FFT flow charts (or "butterfly diagrams") added to the 2nd printing of the Strang book. These famous diagrams look a bit formidable. On the right is the entire (!) Cooley-Tukey algorithm.

EXAMPLE The steps from $n = 4$ to $m = 2$ are

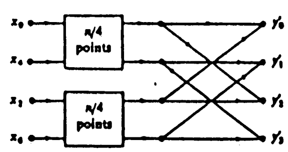
$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \begin{bmatrix} x_0 \\ x_2 \\ x_1 \\ x_3 \end{bmatrix} \to \begin{bmatrix} F_2 x' \\ F_2 x'' \end{bmatrix} \to \begin{bmatrix} y \end{bmatrix}.$$

The Complete FFT and the Butterfly

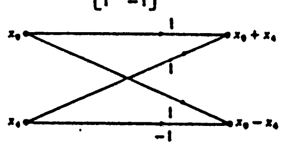
The first step of the FFT changes multiplication by F_n to two multiplications by $F_m = F_{n/2}$. The even-numbered components $(x_0, x_2, \ldots)$ are transformed separately from $(x_1, x_3, \ldots)$. We give a flow graph (added in the second printing) for $n = 8$:



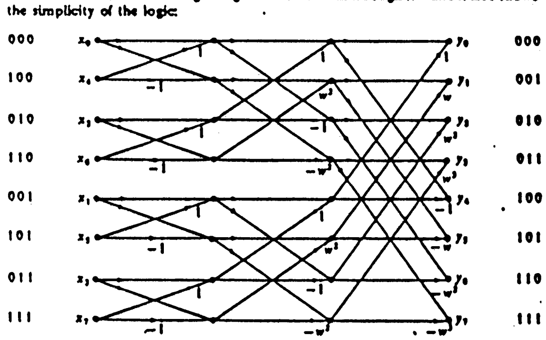
The key idea is *to replace each F_4 box by a similar picture involving two F_2 boxes.* The new factor $w_4 = i$ is the square of the old factor $w = w_8 = e^{2\pi i/8} = (1 + i)/\sqrt{2}$. The top half of the graph changes from F_4 to



Then each of those boxes for $F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is a single butterfly:



By combining the graphs you can see the whole picture. It shows the order that the n x's enter the FFT and the $\log n$ stages that take them through it—and it also shows the simplicity of the logic:



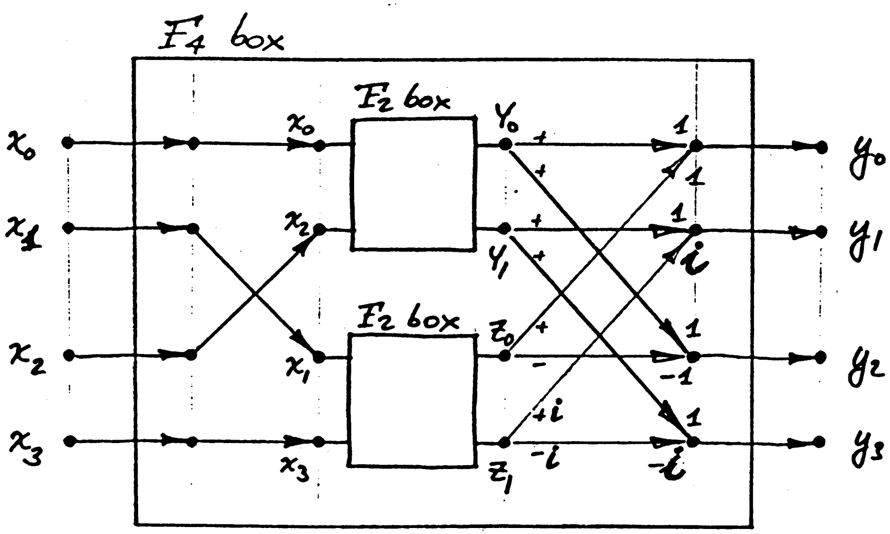
Every stage needs $\frac{1}{2}n$ multiplications so the final count is $\frac{1}{2}n \log n$. There is an amazing rule for the permutation of x's before entering the FFT: Write the subscripts $0, \ldots, 7$ in binary and *reverse the order of their bits.* The subscripts appear in "bit-reversed order" on the left side of the graph. Even numbers come before odd (numbers ending in 0 come before numbers ending in 1) and this is repeated at every stage.

Before the applications, we need to remember what the FFT achieves. It mixes the eight inputs into combinations like $y_0 = x_0 + x_1 + \cdots + x_7$ and $y_1 = x_0 + wx_1 + \cdots + w^7 x_7$. Those two outputs should be traceable on the graph (and $y_2, \ldots, y_7$ are similar). The eight combinations of eight x's are produced without 64 multiplications; 12 will do. The inverse transform F^{-1} just changes w to its conjugate $\bar{w}$—nothing more, except a final division by n. This reorganization has made digital signal processing an overwhelming success. Section 4.2 applied it to convolution; here we apply it to $-u_{xx} - u_{yy} = f$.

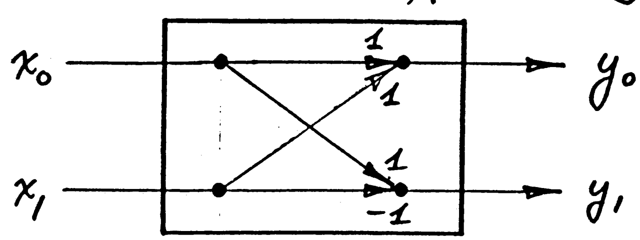
```
subroutine FFT (M)

complex X, U,W, TEMP
common / FDATA / X(1024)

N = 2**M
NV2 = N/2
NM1 = N-1
j = 1

    do 10 i=1,NM1

    if (i.ge.j) go to 5
    TEMP = X(j)
    X(j) = X(i)
    X(i) = TEMP

5   k = NV2
6   if (k.ge.j) go to 10
    j = j-k
    k = k/2
    go to 6

10  j = j+k

    do 40 L=1,M

    LE = 2**L
    LE1 = LE/2
    ang = 3.14159265358979/LE1
    W = CMPLX(cos(ang),sin(ang))
    U = (1.0, 0.0)

    do 30 j=1,LE1

    do 20 i=j,N,LE
    ip = i+LE1
    TEMP = X(ip)*U
    X(ip) = X(i) - TEMP
20  X(i) = X(i) + TEMP

30  U = U*W

40  continue


    return
    end
```

Continuing with this $N=4$ example ...

although matrix notation prefers input on the right, and output on the left, our cause-and-effect thinking prefers just the opposite :



whereas the "F_2 box" is what Gil Strang, in 18.086 book terms "a single butterfly:



Similarly for $N=8$, now using $w = e^{2\pi i/8}$:

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_7 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 & w^4 & w^5 & w^6 & w^7 \\ 1 & w^2 & w^4 & w^6 & w^8 & & & \\ 1 & w^3 & w^6 & w^9 & & & & \\ 1 & w^4 & w^8 & & & & & \\ 1 & w^5 & & & & & & \\ 1 & w^6 & & & & & & \\ 1 & w^7 & & & & & & w^{49} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_7 \end{bmatrix}$$

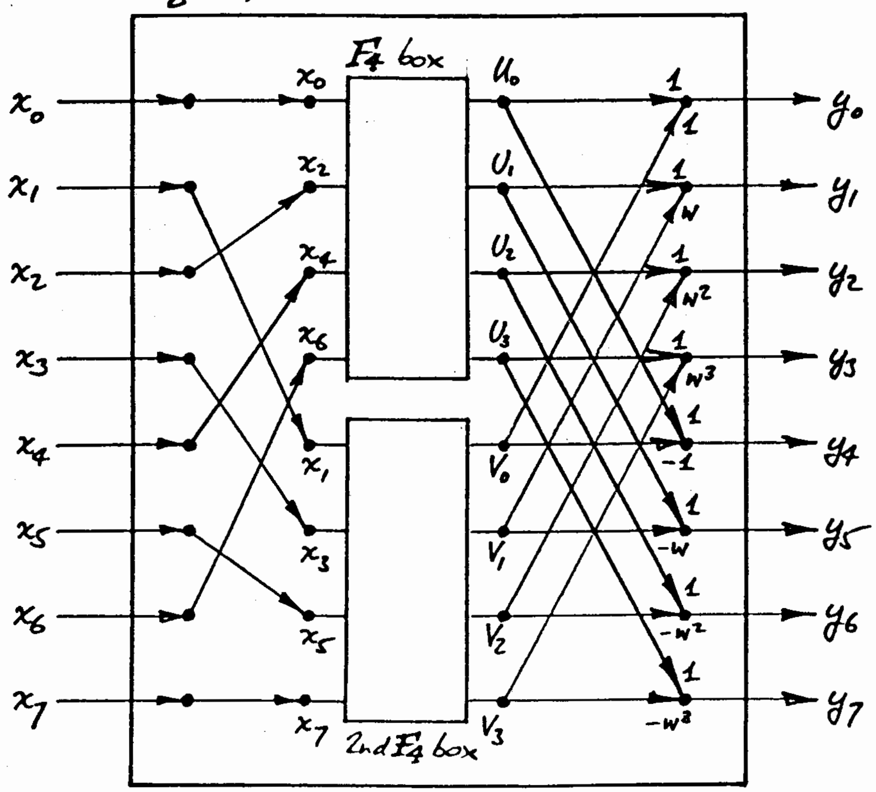
$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^4 & w^8 & w^{12} \\ 1 & w^6 & w^{12} & w^{18} \\ 1 & w^8 & w^{16} & w^{24} \\ 1 & w^{10} & w^{20} & w^{30} \\ 1 & w^{12} & w^{24} & w^{36} \\ 1 & w^{14} & w^{28} & w^{42} \end{bmatrix} \begin{bmatrix} x_0 \\ x_2 \\ x_4 \\ x_6 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 \\ w & w^3 & w^5 & w^7 \\ w^2 & w^6 & w^{10} & w^{14} \\ w^3 & w^9 & w^{15} & w^{21} \\ w^4 & w^{12} & & \\ w^5 & & \cdot & \\ w^6 & & \cdot & \cdot \\ w^7 & & \cdot & \cdot \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_5 \\ x_7 \end{bmatrix}$$

$$= \begin{bmatrix} F_4 \vec{x}_e \\ F_4 \vec{x}_e \end{bmatrix} + \begin{bmatrix} [1, w, w^2, w^3]\, F_4 \vec{x}_{odd} \\ -\text{above} \end{bmatrix}$$

or $\vec{y} = \begin{bmatrix} \vec{U} \\ \vec{U} \end{bmatrix} + [1, w, w^2, w^3, -1, -w, -w^2, -w^3] \begin{bmatrix} \vec{V} \\ \vec{V} \end{bmatrix}$

and <u>this $N=8$ example continues:</u>

F_8 box



It is fascinating to combine all these inner workings into a <u>single</u> diagram:

F_8 box again