PROFESSOR: Today, we're going to be solving a problem from a final exam. And here it is. It's about a matrix, A. 1, 0, 1, 0, 1, 1, 1, 1, 0. And we know that this matrix has two eigenvalues 1 and 2. And we also know that if we do elimination, the first two pivots will be 1 and 1.

And here are two questions about this matrix. The first one is find lambda 3 in d3, the third eigenvalue in the third pivot. And the second one asks you, what is the smallest a33? So if you can change this entry, what is the smallest number that you can put there that will make the matrix A positive semidefinite?

And also, if instead of changing that entry, you do A plus cI, what is the smallest number, c, that will make that matrix a plus cI positive semidefinite? There's also a third part to the question, but we'll get to that later. Why don't you hit pause and work on these two parts? And when you're ready, come back and I'll show you how I did it.

Hi. I hope you managed to do parts A and B. Let's work on it together. Part A. Well, we want to know what the third eigenvalue is. And you know what the first two are. What else do you know about eigenvalues and the matrix? You know that the sum of all the eigenvalues of the matrix is equal to the trace of the matrix.

So lambda 1 plus lambda 2 plus lambda 3 is equal to the trace of the matrix. In this case, you have 1 plus 2 plus lambda 3 equals to the trace. The trace is the sum of the diagonal entries.

So, come over here. The trace is 1 plus 1 plus 0. The trace is equal to 2. So we have 3 plus lambda 3 is equal to 2. So lambda 3 is equal to minus 1. Onto the third pivot.

We don't really want to do elimination. That would take too long. So there must be some trick that we can use. Well, we have the first two pivots, and we want to know the third. Remember, when you do elimination steps, that does not change the determinant of the matrix. And you're left with an upper triangular.

So the determinant of that matrix will be d1 times d2 times d3. And it will still be equal to the determinant of A. I guess there's a small caveat that I should point out. The pivots are not always the diagonal entries. It might be that one of the diagonal entries will be 0. That happens if the matrix is singular.

But here, all my three eigenvalues are non-zero. They are 1, 2, and -1. So that won't happen. So this is actually possible to do. The product of the three pivots will be equal to the determinant of A. And the determinant of A is the product of the eigenvalues. 1 times 2 times -1. So it's equal to -2. 1 times 1 times d3 is equal to -2.

Here is your third pivot, d3. That finishes part A. Is that the result that you got? Let's do part B. What is the smallest a33 entry that would make the matrix positive semidefinite?

Well, first of all, note that A is not positive semidefinite yet. The eigenvalues are 1, 2, and -1. -1 is negative, so the matrix is not positive definite and not even positive semidefinite. Positive semidefinite means that all the eigenvalues will be either positive or 0. That is, non-negative.

So our matrix will be 1, 0, 1, 0, 1, 1, 1, 1. And we're allowed to change this third entry. How do we figure out if this matrix is positive semidefinite or not? Well, I was talking about the eigenvalues. But maybe the easiest way is to do the determinant test.

The determinant of the small one by one left uppermost matrix is 1. The determinant of the two by two upper leftmost matrix is 1 times 1 minus 0 times 0. 1, also positive. So we need to check that the determinant of the whole matrix will also be non-negative.

So what is the determinant of this matrix? It is equal to the three by three matrix. So do you know how to do it quickly? There's this way that only works for three by three and not for bigger. Which is, the determinant will be 1 times 1 times a33. plus 0 times 1 times 1. That's 0. Plus 1 times 0 times 1. That's 0 again. Minus 1 times 1 times 1 minus 1 times 1 times 1 minus a33 times 0 times 0. That's 0.

So this is the determinant. It's a33 minus 2. And I want it to be greater than or equal to 0. This will guarantee that the matrix is positive semidefinite. So a33 must be bigger than or equal to 2. The smallest value for a33 that will make the matrix positive semidefinite is 2.

There's another part to the question still, which is what is the smallest c that will make the matrix A plus cI positive semidefinite? How should we do this? The quickest way is to do the eigenvalue test. A has eigenvalues 1, 2, and -1.

So A plus cI has eigenvalues. Well, you're just adding cI to the matrix. And in this particular case, you should know by now that that keeps the eigenvectors the same and adds the number c to each of the eigenvalues.

And I want each one of these to be non-negative. For that to be true, I have to have c greater than or equal to 1. c greater than or equal to 1. So the smallest value that c can take that will make the matrix A positive semidefinite is 1.

That solves parts A and B of this question. There is a part C to this question. Let me show it to you. It's says, starting with one of these three vectors, 3, 0, 0, 0, 3, 0, or 0, 0, 3, and with uk plus 1 equals to 1/2A times uk, what is the limit behavior of uk as k goes to infinity?

I've written the matrix 1/2 of A here for your convenience. And now, you can hit pause and work on it. And when you're ready, we'll get back and solve it together. I hope you managed to solve this one. Now let's do it together.

Well, if you've noticed the matrix 1/2A is a Markov matrix. So there are all these results about Markov matrices in steady states and so on. Usually, Markov matrices have a unique steady state, but that is only true when there are no non-zero entries. But here, there are. So we can't guarantee that there's a unique steady state.

What we can do is look at the eigenvalues and see if this is still true nonetheless. What are the eigenvalues of A? Of 1/2A? Well, if you remember from part A, the eigenvalues of A were 1, 2, and -1.

So the eigenvalues of 1/2A-- taking a multiple does not change the eigenvector. But it changes the eigenvalue by the same multiple. It would be 1/2, 2 divided by 2 is 1, and minus 1/2.

So here are the eigenvalues of A. And there's only one eigenvalue that has absolute value equal to 1. So you actually still get a unique steady state vector. So everything is fine. We can proceed as usual. And the usual procedure is you find the eigenvector corresponding to that eigenvalue, 1. And that will be the limit behavior as k goes to infinity of uk.

So what is the eigenvector corresponding to 1? Eigenvector. Well, you already know how to do this, so I will just write the solution. It is 1, 1, 1. That means that uk, as k goes to infinity, will converge to some appropriate multiple of this eigenvector 1, 1, 1.

How do you know which multiple to use? Well, as usual in Markov matrices, when you do an iteration of the process, when you do uk plus 1 is equal to 1/2A times uk, that does not change the sum of the entries of the vector uk. So whatever the sum was here, it will still be the same here.

If you go all the way back and you start with u0, whatever the sum of the entries was here, that's what it will be all the way through u1, u2, u3, and so on, all the way to the steady state, u infinity. So whatever the multiple of 1, 1, 1, it has to have the sum of these entries add up to 3.

Well, that's already there. We already happened to pick the correct eigenvector, so that's very convenient. The correct multiple is simply the vector 1, 1, 1. So the limit behavior of uk as k goes to infinity is u infinity equal to 1, 1, 1. We're done. Thank you.

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