

MARTINA Hi. Welcome to recitation.

BALAGOVIC:

Today's this problem is about change of basis. It says the vector space of polynomials in x of degree up to 2 has a basis $1, x,$ and x squared. That's the obvious basis that you would write for that vector space.

But today we're going to consider another basis, $w_1, w_2,$ and w_3 . And we don't know what $w_1, w_2,$ and w_3 are explicitly. What we know is that their values at x equals minus 1, 0, and 1 are given by this table here. So there are $1\ 0\ 0, 0\ 1\ 0,$ and $0\ 0\ 1$.

We're asked to do the following. We're asked to express this polynomial. So y of x is minus x plus 5 in this basis, w_1, w_2, w_3 . We're asked to find the change of basis matrices between these two bases, $1, x, x$ squared, and w_1, w_2, w_3 . And finally, we're asked to find the matrix of taking derivatives, which is a linear map on this space, in both of these basis.

And let me give you an extra level of challenge, which is to try to do as much of this as possible without explicitly finding $w_1, w_2,$ and w_3 . I'll let you think about the problem and then you can come back and compare your solution with mine.

Hi. Welcome back. So to start with the problem a, we need to find coefficients $\alpha,$ $\theta,$ and γ . So that y of x is expressed through, with this coefficient, in this new basis, $w_1, w_2,$ and w_3 .

now, one way to do that would be to look at this table of values, explicitly find $w_1, w_2,$ and w_3 , so a quadratic polynomial is-- all the information we need about it is in values at three points. So say w_1 is a_1 plus b times x plus c times x squared. Find $a,$ $b,$ and c . Find w_1, w_2, w_3 explicitly, and then go back to this system and try to find $\alpha,$ β and γ .

However, there's a trick. Let's try to see if we can do it without finding $w_1, w_2,$ and

w_3 explicitly. So let me try to see what are the values of y at these points. So y is $x^2 - x + 5$. So the values are 6, 5, and 4.

And let me, instead of considering this equation, let me evaluate it at $x = -1$, $x = 0$, and $x = 1$. What I get through this is that w is what is -1 . Actually, let me write this-- I get w with -1 , which is a number, equals α times w_1 is -1 , which is a number, plus β times w_2 is -1 , plus γ times w_3 at -1 .

And similarly at 0, and similarly at 1. And now let me think of this as a linear system that has unknowns α , β , and γ coefficients, these values here at -1 , 0, and 1. And the right-hand side, well what's written here on the left-hand side. y at -1 , y at 0, and y at 1.

If you write this in a matrix and read off coefficients from there, you get the following system. So this is the matrix of the system read off from here. These are the unknowns. And these are the values of the right-hand side. And I hope you'll agree that this is a very easy system to solve. We just get α is 6, β is 5, and γ is 4.

So the solution to the first part is y equals $6w_1 + 5w_2 + 4w_3$. And let's notice another thing. No matter what values we put here, this matrix is always going to stay the same. It's only the right-hand side that's going to change. So if we're given any other polynomial now to express in a basis w_1 , w_2 , and w_3 , we don't have to do any thinking. We don't have to do any computations. What we do is go back to our table at the beginning and just read off, let's go back to the table, and just read these values. So in this case, y is 6 times w_1 , 5 times w_2 , and 3 times w_3 .

And that's already a hint to solving the b part, which is find the change of basis matrices between $1, x, x^2$ and w_1, w_2, w_3 . Change of basis matrices means expressing one basis in terms of another. So as a part of the problem, we will have to express $1, x$, and x^2 in terms of w_1, w_2, w_3 . So let's just find their values at these three points.

One is a constant, it just takes value 1 everywhere. x takes value minus 1 at minus 1, 0 at 0, and 1 at 1. And x squared takes values 1 0 and 1 at minus 1 0 and 1. And from this we can already conclude the part b here, we can conclude that 1 equals w_1 plus w_2 plus w_3 . That x equals minus w_1 plus w_3 . And that x squared equals w_1 plus w_3 .

And from this, we can immediately write one change of basis matrix. Namely, since we know how to express 1, x , and x squared in terms of w_1 , w_2 , and w_3 , we can just copy this information over to this matrix, getting $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

So which matrix is this? This is a matrix, so we have 1, x , and x squared expressed in terms of w_1 , w_2 , and w_3 . So if we feed this matrix something expressed in the basis 1, x , and x squared, say a , b , and c , what is going to throw out is the same polynomial expressed in this basis here, w_1 , w_2 , and w_3 .

So I'm going to just write that this is a matrix of this basis change.

How do we get the other one? Well very easy. We know it's just the inverse of A . So this is going to be the matrix that takes something written in this basis and transfers it to this basis. I'm not going to calculate the inverse of a matrix in front of you.

Instead I'm going to consult my oracle. Sorry about that. And my oracle says that the inverse should be $\begin{pmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{pmatrix}$. And that solves the b part.

Let's go into the c part. The c part required us to find a matrix of taking derivatives, which is a linear map in the space of polynomials, in both of these basis. So let's first do the 1, x , x squared basis because that one's easier. I'm going to call it D_x . So I'm going to work in basis 1, x , x squared. And what I want to express is the transformation of taking derivatives.

So here I'm going to write the vector to which taking derivatives maps the polynomial 1, which is 0. And that this expressed in the basis 1, x , x squared. In the second column I'm going to write x prime, the vector to which D_x send to vector x , and that's equal to 1, which expressed in this basis is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. And here I'm going to

write x squared prime, which is $2x$, which expressed in this basis is just x .

That one was easy. For the other one, well we could calculate w_1 , w_2 , w_3 explicitly, take the derivatives, go back to the table and repeat the procedure that we did already. So expressing these derivatives in terms of w_1 , w_2 , w_3 , and that's a lot of work. But we pretty much already did most of this work. So we know how to take derivatives in this basis, and we know how to go between these two basis.

So if we want to take a derivative of something written in the basis w_1 , w_2 , w_3 , well let's first write this something in basis 1 , x , x squared. Then let's take a derivative of it. And then let's write it back in the original basis that we want. So it's multiplication of three matrices. We have all three-- matrix multiplication is easy. And my oracle, again, says that this should be minus $\frac{3}{2}$, $\frac{2}{2}$, minus $\frac{1}{2}$, minus $\frac{1}{2}$, 0 , $\frac{1}{2}$, and $\frac{1}{2}$, minus $\frac{2}{2}$, $\frac{3}{2}$. And that solves the problem.

Now, one thing that I would want to discuss in the end is how did you do with respect to my challenge, which was let's try to do as much of it as possible without finding w_1 , w_2 , and w_3 explicitly. And it seems like we did really well. There's nowhere on the board written w_1 equals, w_2 equals, w_3 equals.

But is it really so? It's not. We calculated the matrix of A inverse here. And what this really means is that w_1 , w_2 , and w_3 written in the basis 1 , x , and x squared are as follows. w_1 is minus $\frac{1}{2}x$ plus $\frac{1}{2}x$ squared. w_2 is 1 minus x squared. And w_3 is $\frac{1}{2}x$ plus $\frac{1}{2}x$ squared. So you can check your work with the help of this matrix in case you did find w_1 , w_2 , and w_3 explicitly. And that's all I wanted to say today.