SOLUTION SET II FOR 18.075–FALL 2004

10. Functions of a Complex Variable

10.5. **Line Integrals of Complex Functions.** .

33. (a) Use the definition (68) to calculate directly the integral $\oint_C z\ dz$, where C is the unit circle $x = \cos t$, $y = \sin t$.

(b) Use the definition (72) to calculate directly the integral $\oint_C \log z\ dz$, where C is the unit circle $r = 1$, taking the principal value of the logarithm.

(c) Obtain the results of part (a) and (b) by appropriately dealing with the functions $F_1(z) = \frac{z^2}{2}$ and $F_2(z) = z \log z - z$.

Solution. (a) We have $x = \cos t$ and $y = \sin t$ along C. So $dx = -\sin t\ dt$ and $dy = \cos t\ dt$ along C. Then by (68), we have

$$\begin{aligned}
\oint_C z\ dz &= \oint_C (x+iy)(dx+idy) \\
&= \int_0^{2\pi} (\cos t + i \sin t)(-\sin t\ dt + i \cos t\ dt) \\
&= \int_0^{2\pi} (\cos t(-\sin t) - \sin t \cos t)\ dt + i \int_0^{2\pi} (-\sin^2 t + \cos^2 t)\ dt \\
&= -\int_0^{2\pi} \sin 2t\ dt - i \int_0^{2\pi} \cos 2t\ dt \\
&= 0.
\end{aligned}$$

(b) We have that $r = 1$, $dr = 0$ and $log(r, \theta) = i\theta$ along C. Then by (72), we get

$$\begin{aligned}
\oint_C \log z\ dz &= \oint_C (i\theta)(ie^{i\theta} d\theta) \\
&= -\int_0^{2\pi} \theta \cos\theta\ d\theta - i \int_0^{2\pi} \theta \sin\theta\ d\theta \\
&= 0 - i(-2\pi) \\
&= 2\pi i.
\end{aligned}$$

(c) Clearly, we have $F_1'(z) = z$ on the complex plane. So $\oint_C z\ dz = F_1(1) - F_1(1) = 0$. We choose $\log z$ to be the principal branch of logarithm defined on the complex plane with nonnegative real numbers removed. Then $F_2(z)$ is also defined on this simply connected

Date: September 20, 2002.

open set. And we have $F_2'(z) = logz$ on this simply connected open set. For small positive t, let C_t be the arc of C connecting e^{it} and $e^{i(2\pi-t)}$ in this open set. Then

$$\begin{aligned}\oint_C \log z\ dz &= \lim_{t\to 0}\int_{C_t} \log z\ dz \\ &= \lim_{t\to 0}(F_2(e^{i(2\pi-t)}) - F_2(e^{it})) \\ &= (2\pi i - 1) - (-1) \\ &= 2\pi i.\end{aligned}$$

34. (a) Show that the value of the integral

$$\int_{-1}^{1} \frac{z+1}{z^2}\ dz$$

is $-2-\pi i$ if the path is the *upper* half of the circle $r = 1$. [Write $z = e^{i\theta}$, where θ varies from π to 0, or from $(2k+1)\pi$ to $2k\pi$, where k is any integer.]

(b) Show (also by direct integration) that the value is $-2+\pi i$ if the path is the *lower* half of the circle.

(c) Obtain the results of parts (a) and (b) by appropriately dealing with the function $F(z) = \log z - z^{-1}$.

Solution. (a) Parameterize c_u, the upper half of the circle, by $z = e^{i\theta}$, where $0 \le \theta \le \pi$. Then $dz = ie^{i\theta}d\theta$. Substituting these into (1), we get:

$$\begin{aligned}\int_{c_u} \frac{z+1}{z^2}\ dz &= \int_{\pi}^{0} \frac{e^{i\theta}+1}{(e^{i\theta})^2}(ie^{i\theta})\ d\theta \\ &= -\int_0^{\pi}(i + ie^{-i\theta})\ d\theta \\ &= -\pi i + e^{-i\theta}|_0^{\pi} \\ &= -\pi i + (-1-1) \\ &= -2-\pi i.\end{aligned}$$

(b) Parameterize c_l, the lower half of the circle, by $z = e^{\theta}$, where $-\pi \le \theta \le 0$. Then $dz = ie^{i\theta}d\theta$, and, hence,

$$\begin{aligned}\int_{c_l} \frac{z+1}{z^2}\ dz &= \int_{-\pi}^{0} \frac{e^{i\theta}+1}{(e^{i\theta})^2}(ie^{i\theta})\ d\theta \\ &= \int_{-\pi}^{0}(i + ie^{-i\theta})\ d\theta \\ &= \pi i - e^{i\theta}|_{-\pi}^{0} \\ &= \pi i - (1-(-1)) \\ &= -2+\pi i.\end{aligned}$$

(c) Let $F_u(z)$ be the branch of $F(z)$ defined on $\mathbf{C} - i\mathbf{R}_{\leq 0}$ with $F_u(1) = -1$. Then $F_u(-1) = \pi i + 1$. Clearly, $F_u'(z) = \frac{z+1}{z^2}$ on $\mathbf{C} - i\mathbf{R}_{\leq 0}$. Since $\mathbf{C} - i\mathbf{R}_{\leq 0}$ is simply connected, and contains c_u, we have

$$\int_{c_u} \frac{z+1}{z^2}\, dz \;=\; F_u(1) - F_u(-1) \;=\; -1 - (\pi i + 1) \;=\; -2 - \pi i.$$

Similarly, let $F_l(z)$ be the branch of $F(z)$ defined on $\mathbf{C} - i\mathbf{R}_{\geq 0}$ with $F_l(1) = -1$. Then$F_l(-1) = -\pi i + 1$. Clearly, $F_l'(z) = \frac{z+1}{z^2}$ on $\mathbf{C} - i\mathbf{R}_{\geq 0}$. Since $\mathbf{C} - i\mathbf{R}_{\leq 0}$ is simply connected, and contains c_l, we have that

$$\int_{c_l} \frac{z+1}{z^2}\, dz \;=\; F_l(1) - F_l(-1) \;=\; -1 - (-\pi i + 1) \;=\; -2 + \pi i.$$

35 (a) Evaluate the integral

$$\oint_C \frac{z+1}{z^2}\, dz,$$

where C is the circle $r = 1$, first by using the results of Problem 34(a,b), second by considering the function $log z - z^{-1}$, and third by using Equation (75) and (77).

(b) Evaluate the integral in part (a) (by any method) when C is the circle $r = a$ $(a > 0)$.

Solution. (a) First, we have that

$$\oint_C \frac{z+1}{z^2}\, dz \;=\; \int_{C_l} \frac{z+1}{z^2}\, dz - \int_{C_u} \frac{z+1}{z^2}\, dz.$$

Thus, by the results of 34(a,b), we have

$$\oint_C \frac{z+1}{z^2}\, dz \;=\; (-2 + \pi i) - (-2 - \pi i) \;=\; 2\pi i.$$

Second, let $F_1(z)$ be the branch of $F(z)$ defined on $\mathbf{C} - \mathbf{R}_{\geq 0}$ with $F_1(-1) = -1 - \pi i$. For $n \in N_{>0}$, let $p_n = e^{\frac{2\pi i}{n}}$, $q_n = e^{\frac{(2n-2)\pi i}{n}}$, and c_n the arc of C beginning at p_n, passing through -1, and ending at q_n. Then, we have

$$\begin{aligned}
\oint_C \frac{z+1}{z^2}\, dz &= \lim_{n\to\infty} \int_{c_n} \frac{z+1}{z^2}\, dz \\
&= \lim_{n\to\infty} F_1(q_n) - F_1(p_n) \\
&= \lim_{n\to\infty} (-q_n^{-1} + \frac{(2n-2)\pi i}{n}) - (-p_n^{-1} + \frac{2\pi i}{n}) \\
&= \lim_{n\to\infty} [(p_n^{-1} - q_n^{-1}) + \frac{(2n-4)\pi i}{n}] \\
&= (1-1) + 2\pi i \\
&= 2\pi i
\end{aligned}$$

Third, we have $\frac{z+1}{z^2} = z^{-1} + z^{-2}$. Hence,

$$\oint_C \frac{z+1}{z^2}\, dz \;=\; \oint_C z^{-1}\, dz + \oint_C z^{-2}\, dz.$$

By (75) and (77) in the text, we have $\oint_C z^{-1}\, dz = 2\pi i$ and $\oint_C z^{-2}\, dz = 0$. Thus,

$$\oint_C \frac{z+1}{z^2}\, dz \;=\; 2\pi i.$$

(b) By (75) and (77) in the text, we have $\oint_C z^{-1}\, dz = 2\pi i$ and $\oint_C z^{-2}\, dz = 0$. Therefore,

$$\oint_C \frac{z+1}{z^2}\, dz \;=\; \oint_C z^{-1}\, dz + \oint_C z^{-2}\, dz \;=\; 2\pi i.$$

36 Evaluate the integral

$$\oint_C \overline{z}\, dz,$$

where C is the unit circle $r = 1$ and also, more generally, when C is the circle $r = a$ ($a > 0$).

Solution. Let C be the circle $r = a$. Then we have $x = a\cos t$ and $y = a\sin t$ along C. So $dx = -a\sin t dt$ and $dy = a\cos t dt$ along C. Then by (68), we have

$$\begin{aligned}
\oint_C \overline{z}\, dz &= \oint_C (x - iy)(dx + idy) \\
&= \int_0^{2\pi} (a\cos t - ia\sin t)(-a\sin t\; dt + ia\cos t\; dt) \\
&= \int_0^{2\pi} (-a^2\cos t\sin t + a^2\sin t\cos t)\; dt + i\int_0^{2\pi} (a^2\sin^2 t + a^2\cos^2 t)\; dt \\
&= 2\pi a^2 i.
\end{aligned}$$

In particular, if $r = 1$ then

$$\oint_C \overline{z}\, dz = 2\pi i.$$

37 Proceed as in Problem 36 with the integral

$$\oint_C (|z| - e^z \sin z^2)\; dz.$$

Solution. Let C be the circle $r = a$. Since the function $e^z \sin z^2$ is analytic inside and on the closed curve C, Cauchy's integral theorem implies that

$$\oint_C e^z \sin z^2\; dz = 0.$$

Therefore, since $|z| = a$ along C, we get

$$\oint_C (|z| - e^z \sin z^2)\; dz = \oint_C |z|\; dz = a\oint_C dz = 0.$$

38 (a) Prove the the integral

$$\int_{-1}^{2} \frac{dz}{z^2}$$

is independent of the path, so long as that path does not pass through the origin. By integrating along any convenient path (say, around a semicircle and thence along the real axis) show that the value of the integral is $-\frac{3}{2}$.

(b) Show that the real integral

$$\int_{-1}^{2} \frac{dx}{x^2}$$

does not exist, but that the value given by formal substitution of limits in the indefinite integral agrees with that obtained in part (a). (Notice that, in spite of this fact, the integrand is never negative!)

Solution. (a) Let P_1 and P_2 be any two paths from -1 to 2, that do not pass through 0. Then P_1 followed by the inverse of P_2 forms a closed path c from -1 to -1, which does not pass through 0. So we have

$$\int_{P_1} \frac{dz}{z^2} - \int_{P_2} \frac{dz}{z^2} \ = \ \int_{P_1} \frac{dz}{z^2} + \int_{-P_2} \frac{dz}{z^2} \ = \ \oint_{c} \frac{dz}{z^2} \ = \ 0,$$

where we used (77) in the last step. This shows $\int_{-1}^{2} \frac{dz}{z^2}$ is independent of the path so long as the path does not pass through the origin.

Let c_1 be the path given by $z = e^{i\theta}$, where $-\pi \leq \theta \leq 0$, and c_2 the path given by $z = t + 0i$, where $1 \leq t \leq 2$. Then

$$\begin{aligned}
\int_{-1}^{2} \frac{dz}{z^2} &= \int_{c_1} \frac{dz}{z^2} + \int_{c_2} \frac{dz}{z^2} \\
&= \int_{-\pi}^{0} e^{-2i\theta}(ie^{i\theta})\ d\theta + \int_{1}^{2} t^{-2}\ dt \\
&= -e^{i\theta}|_{-\pi}^{0} - t^{-1}|_{1}^{2} \\
&= -2 + \frac{1}{2} \\
&= -\frac{3}{2}.
\end{aligned}$$

(b) Suppose $\int_{-1}^{2} \frac{dx}{x^2}$ exits. Then

$$\int_{-1}^{2} \frac{dx}{x^2} \geq \int_{0}^{2} \frac{dx}{x^2} \ = \ \lim_{\epsilon \to 0} \int_{\epsilon}^{2} \frac{dx}{x^2} \ = \ \lim_{\epsilon \to 0} -\frac{1}{x}|_{\epsilon}^{2} \ = \ \infty.$$

This is a contradiction. So $\int_{-1}^{2} \frac{dx}{x^2}$ does not exist as real integral. But, by a symbolic calculation, we have

$$\int_{-1}^{2} \frac{dx}{x^2} \ = \ -\frac{1}{x}|_{-1}^{2} \ = \ -\frac{1}{2} - 1 \ = \ -\frac{3}{2}.$$

10.6. **Cauchy's Integral Formula.** .

43. If $F(z) = (z+6)/(z^2-4)$, show that the integral $\oint_C F(z)\, dz$ is 0 if C is the circle $x^2+y^2=1$, is $4\pi i$ if C is the circle $(x-2)^2+y^2=1$, and is $-2\pi i$ if C is the circle $(x+2)^2+y^2=1$.

Solution. By writing

$$F(z) = \frac{z+6}{(z-2)(z+2)},$$

we see that F is analytic at all points except for -2 and 2. These points are outside the circle $x^2+y^2=1$, and therefore the integral is 0.

Also, $F(z) = g(z)/(z-2)$, where $g(z) = (z+6)/(z+2)$. In the interior of the circle $(x-2)^2+y^2=1$, $g(z)$ is analytic. From Cauchy's Integral Formula,

$$\oint_C F(z)\, dz \ = \ \oint_C \frac{g(z)}{z-2}\, dz \ = \ 2\pi i g(2) \ = \ 4\pi.$$

44. *A mean-value theorem.* Let z_0 denote a point in a region $\mathcal{R}$ where $f(z)$ is analytic, and let C denote any circle, with center at z_0, which lies inside $\mathcal{R}$. By writing $\alpha = z_0 + ae^{i\phi}$ in Cauchy's integral formula (85), show that

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + ae^{i\phi})\, d\phi,$$

and deduce that *the value of an analytic function at any point z_0 is the average of its values on any circle, with z_0 as its center, which lies inside the region of analyticity.*

Solution. According to Cauchy's integral formula,

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(\alpha)}{\alpha - z_0}\, d\alpha,$$

where $\alpha = z_0 + a\, e^{i\phi}$ and ϕ lies in $[0, 2\pi)$. It follows that $d\alpha = a\, d(e^{i\phi}) = ia\, e^{i\phi}\, d\phi$ and Cauchy's formula becomes

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + a\, e^{i\phi})}{ae^{i\phi}}\, iae^{i\phi}\, d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + ae^{i\phi})\, d\phi, \end{aligned}$$

which is the desired formula.