18.075

In-class

Solutions to Practice Test 1

(I) $w = z^{m/n} \iff z = w^{n/m}$ (all possible values)

Recall : If $w = |w| \cdot e^{i\theta_p}$ (say $0 \le \theta_p < 2\pi$), then

$$w^{n/m} = \underbrace{|w|^{n/m}}_{>0 \text{ (positive)}} \cdot e^{i(\theta_p + 2k\pi)\frac{n}{m}}, \quad k = 0, 1, 2, \dots, m-1$$

(n, m : w/ no commm factor)

Hence, $z^{4/3} = 1+i \iff z = (1+i)^{3/4}$

We need to find all possible values of $(1+i)^{3/4}$ ($n=3$, $m=4$)

$$1+i = \underbrace{|1+i|}_{r} \cdot e^{i\theta_p} = \sqrt{1+1} \cdot e^{i\pi/4} = \sqrt{2}\, e^{i\pi/4}; \qquad r = \sqrt{2}, \ \theta_p = \pi/4.$$

$$(1+i)^{3/4} = (\sqrt{2})^{3/4} \cdot e^{i\left(\frac{\pi}{4} + 2k\pi\right)\frac{3}{4}} = \underbrace{2^{3/8}}_{>0} \cdot e^{i\left(\frac{\pi}{4}+2k\pi\right)\cdot\frac{3}{4}} = z_k, \quad k = 0, 1, 2, 3.$$

$k=0$: $z_0 = 2^{3/8} \cdot e^{i\frac{3\pi}{16}}$

$k=1$: $z_1 = 2^{3/8} \cdot e^{i\frac{9\pi}{4}\cdot 3/4} = 2^{3/8} \cdot e^{i\frac{27\pi}{16}}$

$k=2$: $z_2 = 2^{3/8} \cdot e^{i\frac{17\pi}{4}\cdot 3/4} = 2^{3/8} \cdot e^{i\frac{51\pi}{16}}$

$k=3$: $z_3 = 2^{3/8} \cdot e^{i\frac{75\pi}{16}}$.

(II) (1), (2) $v(x,y) = 4xy + x + y$

Check the Cauchy - Riemann equations:

$$\underbrace{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} = 4x+1 \iff u(x,y) = 2x^2 + x + \underbrace{C(y)}_{\text{real fcn of } y}$$

$$C'(y) = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -4y-1 \iff C(y) = -2y^2 - y + \underbrace{K}_{\text{real const.}}$$

Hence, $u(x,y) = 2x^2 + x - 2y^2 - y + K$

So, yes, $v(x,y)$ can be the imaginary part of analytic $f(z) = u + iv$.

Alternatively, for part (1), simply check that $v(x,y)$ satisfies Laplace's equation:

$$\left.\begin{array}{ll} \dfrac{\partial v}{\partial x} = 4y+1, & \dfrac{\partial^2 v}{\partial x^2} = 0 \\ \dfrac{\partial v}{\partial y} = 4x+1, & \dfrac{\partial^2 v}{\partial y^2} = 0 \end{array}\right\} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

So, v can be the imag. part of analytic $f(z)$
($\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial^2 v}{\partial x^2}, \frac{\partial^2 v}{\partial y^2}$ are continuous)

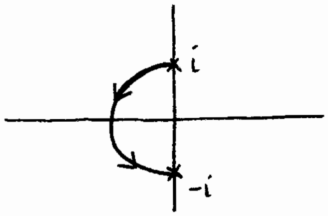
(3) $f(z) = u + iv = (2x^2 + x - 2y^2 - y + K) + i(4xy + x + y)$

$$= x + iy + (-y + ix) + (2x^2 - 2y^2 + i4xy) + K$$

$$= \underbrace{x+iy}_{z} + i\underbrace{(x+iy)}_{z} + 2\underbrace{(x^2 - y^2 + i2xy)}_{z^2} + K = (1+i)z + 2z^2 + K$$

$(z = x + iy)$

III



$$I = \int_C dz \, \frac{z^2-2}{z^3} = \int_C dz \cdot \left(\frac{1}{z} - \frac{2}{z^3} \right)$$

- $$\int_C \frac{dz}{z} = \int_{\pi/2}^{3\pi/2} \frac{d(e^{i\theta})}{e^{i\theta}} = \int_{\pi/2}^{3\pi/2} \frac{i e^{i\theta}\, d\theta}{e^{i\theta}} = i\pi$$

- $$\int_C dz \, \frac{2}{z^3} = 2 \int_{\pi/2}^{3\pi/2} \frac{d(e^{i\theta})}{e^{3i\theta}} = 2i \int_{\pi/2}^{3\pi/2} \frac{d\theta}{e^{2i\theta}} = 2i \left. \frac{e^{-2i\theta}}{-2i} \right|_{\pi/2}^{3\pi/2}$$

$$= -1 \cdot \left(e^{-2i\, 3\pi/2} - e^{-2i\, \pi/2} \right) = -\left(-1 + 1 \right) = 0.$$

$$I = i\pi$$

IV ① $$f(z) = \frac{z}{(1-z)(z+3)} = \frac{A}{z-1} + \frac{B}{z+3}$$

$$A = \lim_{z \to 1} \left[f(z) \cdot (z-1) \right] = \lim_{z \to 1} \left(\frac{-z}{z+3} \right) = \frac{-1}{4}$$

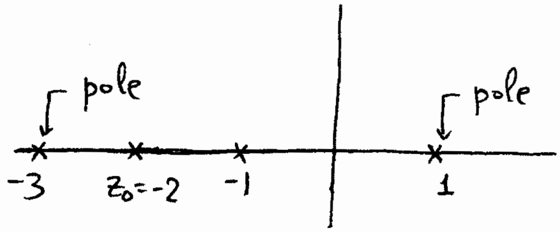
$$B = \lim_{z \to -3} \left[(z+3) f(z) \right] = \lim_{z \to -3} \left(\frac{z}{1-z} \right) = \frac{-3}{4}$$

$$f(z) = \frac{-1}{4} \frac{1}{z-1} - \frac{3}{4} \frac{1}{z+3}$$

② Isolated singularities @ $z = 1, -3$.

These are simple poles, because $(z-1) f(z)$ is analytic @ $z = 1$ with $\lim_{z \to 1} [(z-1) f(z)] \neq 0$ and $(z+3) f(z)$ is analytic @ $z = -3$ with $\lim_{z \to 3} [(z+3) f(z)] \neq 0$.

③



The given function is analytic for $0 \le |z+2| < 1$, $1 < |z+2| < 3$

and for $3 < |z+2| < \infty$.

(i) No, because $f(z)$ has a pole @ $z=1$ inside $1 < |z+2| < 4$.

(ii) Yes, because $f(z)$ is analytic in $3 < |z+2|$.

(iii) Yes, because $f(z)$ is analytic in $1 < |z+2| < 3$

④ $$f(z) = -\frac{1}{4}\frac{1}{z-1} - \frac{3}{4}\frac{1}{(z-1)+4}$$

$$= -\frac{1}{4}\frac{1}{z-1} - \frac{3}{4}\cdot\frac{1}{4}\frac{1}{1+\underbrace{\frac{z-1}{4}}_{-\lambda}}$$

$$= -\frac{1}{4}\frac{1}{z-1} - \frac{3}{16}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{4}\right)^n, \quad 0< |z-1| < 4$$

(hence, it also converges in the smaller disk $0<|z-1|<2$)

(V) ① $$f(z) = \frac{1}{(z^2-z-2)^2}$$

Possible singularities: $z^2-z-2=0 \Rightarrow z = \frac{1\pm 3}{2} = \begin{cases} 2 \\ -1 \end{cases}$

$$f(z) = \frac{1}{(z-2)^2(z+1)^2}.$$

Note that $\quad (z-2)^2 f(z) = \dfrac{1}{(z+1)^2}$: analytic and $\neq 0$ @ $z=2$

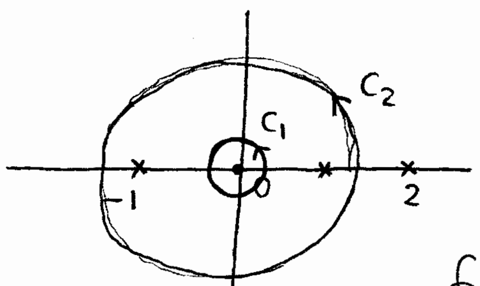
$\quad (z+1)^2 f(z) = \dfrac{1}{(z-2)^2}$: analytic and $\neq 0$ @ $z=-1$.

Hence, $f(z)$ has <u>double poles</u> @ $z=-1, 2$

Residues: $\quad \text{Res}(-1) = \dfrac{d}{dz}\left[\dfrac{1}{(z-2)^2}\right]\Big|_{z=-1} = \dfrac{-2}{(z-2)^3}\Big|_{z=-1} = \dfrac{-2}{(-3)^3} = \dfrac{2}{27}$

$\quad \text{Res}(2) = \dfrac{d}{dz}\left[\dfrac{1}{(z+1)^2}\right]\Big|_{z=2} = \dfrac{-2}{(z+1)^3}\Big|_{z=2} = \dfrac{-2}{3^3} = \dfrac{-2}{27}$.

②



C_1 : radius $1/4$, encloses <u>no</u> singularity

C_2 : radius $5/4$, encloses $z_0 = -1$ only

C_3 : radius 4, encloses <u>both</u> $z_0 = -1, 2$

- $\oint_{C_1} dz\, f(z) = 0$
- $\oint_{C_2} dz\, f(z) = 2\pi i \cdot \text{Res}(-1) = \dfrac{4\pi i}{27}$
- $\oint_{C_3} dz\, f(z) = 2\pi i\, \left[\text{Res}(-1) + \text{Res}(2)\right] = 2\pi i \left(\frac{2}{27} - 2/27\right) = 0$

VI ① With $w = z^{1/2}$, expand

$$\cos w = 1 - w^2/2! + w^4/4! - \frac{w^6}{6!} + \dots + (-1)^n \frac{w^{2n}}{(2n)!} + \dots$$

$$\Rightarrow \cos z^{1/2} = 1 - \frac{(z^{1/2})^2}{2!} + \frac{(z^{1/2})^4}{4!} - \frac{(z^{1/2})^6}{6!} + \dots + (-1)^n \frac{(z^{1/2})^{2n}}{(2n)!} + \dots$$

$$= 1 - \frac{z}{2!} + \frac{z^2}{4!} - \frac{z^3}{6!} + \dots + (-1)^n \frac{z^n}{(2n)!} + \dots$$

This is a Taylor series!
(Only nonnegative integral powers of z appear!)

Hence, $z=0$ is NOT a singular point.

$f(z)$ is analytic at $z=0$

(2) $f(z) = \dfrac{\cos z - 1}{\sinh z - z}$

Expand:

$$\cos z = 1 - z^2/2! + z^4/4! - \dots + (-1)^n \frac{z^{2n}}{(2n)!} + \dots$$

$$\sinh z = z + z^3/3! + z^5/5! + \dots + \frac{z^{2n+1}}{(2n+1)!} + \dots$$

So,

$$\cos z - 1 = -z^2 \left[\frac{1}{2!} - \frac{z^2}{4!} + \dots + (-1)^{n+1} \frac{z^{2n-2}}{(2n)!} + \dots \right]$$

$$\sinh z - z = z^3 \left[\frac{1}{3!} + z^2/5! + \dots + \frac{z^{2n-2}}{(2n+1)!} + \dots \right]$$

$$\Rightarrow f(z) = \frac{\cos z - 1}{\sinh z - z} = -\underbrace{\frac{1}{z}}_{\text{has a simple pole @ } z=0} \underbrace{\left(\frac{\frac{1}{2!} - \frac{z^2}{4!} + \dots + (-1)^{n+1} \frac{z^{2n-2}}{(2n)!}}{1/3! + z^2/5! + \dots + \frac{z^{2n-2}}{(2n+1)!} + \dots} \right)}_{\text{T.S.; analytic @ } z=0}$$

Hence, $f(z)$ has a simple pole @ $z_0 = 0$.