Real Integrals

(I) integrals $\int_a^b f(x)\,dx$, $f(x) = \frac{P_M(x)}{Q_N(x)}$ > polynomials $a = 0$ or $-\infty$, $b = \infty$

<u>ex</u> $I = \int_0^\infty \frac{dx}{1+x^2}$ $P = 1$, $Q = 1+x^2$

Method A: $\frac{1}{1+x^2} = \frac{d}{dx}\arctan x$

$$I = \arctan(x)\Big|_0^\infty = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

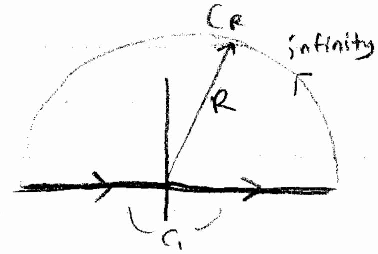
Method B: contour integration: use residue theorem (pay attention to steps)

i. mark singularities, integration path

$f(x) = \frac{1}{1+x^2}$; $x \to z$, $f(z) = \frac{1}{1+z^2}$ Where is $f(z)$ analytic?

everywhere except at $z = \pm i$ (simple poles)

ii. want to close the path



$\frac{1}{1+z^2} = \frac{1}{1+(-z)^2}$: symmetric function

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{1}{2}\int_{C_1} \frac{dz}{1+z^2} \qquad C = C_1 + C_R$$

iii. apply residue theorem

$$\oint_C f(z)dz = 2\pi i \sum \underset{z=i}{\text{Res}}\, f(z) = 2\pi i \frac{1}{2i} = \pi$$

$$= \left(\int_{C_1} + \int_{C_R}\right) f(z)dz = 2I + \int_{C_R} f(z)dz \quad R \to +\infty$$

z on $C_R = Re^{i\theta}$ $\qquad dz = iRe^{i\theta}d\theta$

$$\left|\int_{C_R} f(z)dz\right| = \left|\int_0^{\pi} \frac{1}{1+(Re^{i\theta})^2} \cdot iRe^{i\theta}d\theta\right| \le \int_{\theta=0}^{\pi} \frac{R}{|1+R^2e^{2i\theta}|}d\theta$$

$\downarrow$

$\lim_{R\to\infty} = 0$

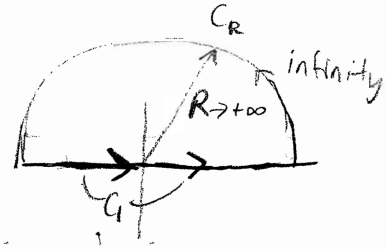
$$\left(\left|\int_a^b G(\theta)d\theta\right| \le \int_a^b |G(\theta)|d\theta\right)$$

G: complex, θ: real

$$2I + \underbrace{\int_{C_R} \frac{1}{1+z^2}dz}_{(R\to\infty)}{}^{\,0} = \pi$$

$$I = \frac{\pi}{2}$$

$$|e^{i\theta}| = 1 \quad \leftarrow \text{real}$$



ii. Want to close the path.

$$\frac{1}{1+z^2} = \frac{1}{1+(-z)^2} \quad \text{: symmetric function}$$

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{1}{2}\int_{C_1} \frac{dz}{1+z^2} \qquad C = C_1 + C_R$$

iii. $\oint_C f(z)dz = 2\pi i \sum_{z=i} \text{Res}\, f(z) = 2\pi i \cdot \frac{1}{2i} = \pi$ apply residue theorem

$$= \left(\int_{C_1} + \int_{C_R}\right) f(z)dz = 2I + \underbrace{\boxed{\int_{C_R} f(z)dz}}_{R \to \infty}$$

z on $C_R = Re^{i\theta}$ $\quad dz = iRe^{i\theta}d\theta$

$$\left|\int_{C_R} f(z)\,dz\right| = \left|\int_0^{\pi} iRe^{i\theta}d\theta \frac{1}{1+(Re^{i\theta})^2}\right| \le \int_{\theta=0}^{\pi} d\theta \frac{R}{|1+R^2e^{2i\theta}|} \xrightarrow{\lim_{R\to\infty}} 0$$

$\left|\int_a^b G(\theta)d\theta\right| \le \int_a^b d\theta\, |G(\theta)|$ $\quad G$: complex θ: real

$$2I + \underbrace{\cancel{\int_{C_R} dz \frac{1}{1+z^2}}}_{(R\to\infty)} \nearrow 0 = \pi$$

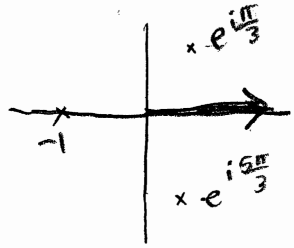
$$I = \frac{\pi}{2}$$

ex $\int_0^{\infty} \frac{dx}{1+x^3}$

i. let $x \to z$ $\quad f(z) = \frac{1}{1+z^3}$, analytic everywhere except at $z^3 = -1 \to z = (-1)^3$

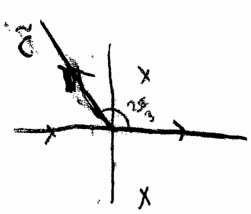
$-1 = re^{i\theta_p} = |e^{i\pi}$ $\quad (-\pi < \theta_p \le \pi)$

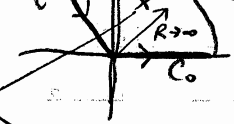
$$z = \left(e^{i\pi + i2k\pi}\right)^{\frac{1}{3}} = e^{\frac{i\pi}{3} + i\frac{2k}{3}\pi} \qquad k = 0, 1, 2$$



ii. $\frac{1}{1+z^3} = \frac{1}{1+(w)^3}$ $\qquad w^3 = z^3 \longrightarrow \begin{cases} w = z \\ w = e^{i\frac{2\pi}{3}}z \\ w = e^{i\frac{4\pi}{3}}z \end{cases}$ ← cube roots of 1

↑ rotation





$\frac{g(z)}{h(z)}$

along $\tilde{C}$, $z: |z| e^{i\frac{2\pi}{3}} = t e^{i\frac{2\pi}{3}}$, $t>0$

$|z| = t$

$C = C_0 + \tilde{C} + C_R$

iii. $\oint_C f(z)\,dz = 2\pi i \frac{1}{(3 \cdot z^2)}\Big|_{z=e^{i\frac{\pi}{3}}} = \frac{2\pi i}{3} e^{-i\frac{2\pi}{3}}$

$\left(\int_{C_0} + \int_{\tilde{C}} + \cancel{\int_{C_R}}^{\,0}\right) f(z)\,dz = 1 - e^{i\frac{2\pi}{3}}$

$dz = dt\, e^{i\frac{2\pi}{3}}$

$\int_{\tilde{C}} f(z)\,dz = -\int_0^\infty e^{i\frac{2\pi}{3}} dt \frac{1}{1+t^3} = -e^{i\frac{2\pi}{3}} I$

$\left(\int_{C_0} + \int_{\tilde{C}}\right) f(z)\,dz = I\left(1 - e^{i\frac{2\pi}{3}}\right)$

real integral

$\left(1 - e^{i\frac{2\pi}{3}}\right) I = \frac{2\pi i}{3} e^{-i\frac{2\pi}{3}} \longrightarrow I = \frac{\pi/3}{\sin \pi/3}$

$e^{i\frac{\pi}{3}}\underbrace{\left(e^{-i\frac{\pi}{3}} - e^{i\frac{\pi}{3}}\right)}_{-2i\sin\frac{\pi}{3}}$

difference of two opposite exponents

Whenever you see $1 -$ phase factor

$$\boxed{I = \int_0^\infty dx \frac{1}{1+x^m} = \frac{\pi/m}{\sin(\pi/m)} \quad m: \text{integer}}$$