8. logarithm: $w = \mathrm{Ln}\, z$ or $\underset{\uparrow \text{complex}}{\mathrm{Log}}\, z$ → "natural log" of z

$w = \mathrm{Ln}\, z \leftrightarrow z = e^{w}$

Suppose $z = x + iy = re^{i\theta_p}$, $\theta_p : 0 \le \theta_p < 2\pi$, $w = u + iv$ u, v, x, y: real

$\mathrm{Arg}\, z = \theta_p$

$$z = e^{w} \rightarrow x + iy = e^{u+iv} = e^{u} e^{iv} = e^{u}(\cos v + i \sin v)$$

$$\rightarrow \begin{cases} x = e^{u} \cos v \\ y = e^{u} \sin v \end{cases} \quad ① \; \begin{matrix} x^2 = e^{2u} \cos^2 v \\ y^2 = e^{2u} \sin^2 v \end{matrix} \quad \underbrace{x^2 + y^2}_{r^2} = e^{2u}(\cos^2 v + \sin^2 v) = e^{2u}$$

$r^2 = e^{2u}$

$r = e^{u}$

$u = \ln r = \mathrm{Re}(w)$

real natural log of r, $r > 0$, $r = |z|$

$$② \; \frac{x}{y} = \frac{\cos v}{\sin v} \rightarrow \tan v = \frac{y}{x} = \frac{r \sin\theta}{r \cos\theta} = \tan\theta$$

$\tan v = \tan\theta$

$\sin v = \sin\theta$

$v = \theta$ $\theta_p : 0 \le \theta_p < 2\pi$

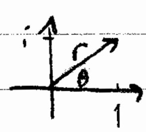
principal value of θ ($\arg z$)

$$w = \mathrm{Ln}\, z = u + iv = \boxed{\ln|z| + i(\underbrace{\theta_p + 2k\pi}_{\theta})} \quad k: \text{integer}$$

$k = 0$: $\mathrm{Ln}\, z$ is principal value of the logarithm

ex $z = 1 + i$, $\mathrm{Ln}\, z = ?$

$|z| = r = \sqrt{2}$, $\theta_p = \frac{\pi}{4}$



← careful! 1st quadrant

$$\boxed{\mathrm{Ln}\, z = \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2k\pi\right)} \quad k: \text{integer}$$

9. generalized power function: $w = f(z) = z^c$ c: complex

(i) $c = n$: positive integer: $z^c = z^n = (e^{Lnz})^n = e^{nLnz} = e^{n(\ln|z| + i(\theta_p + 2k\pi))}$
$= e^{n\ln|z| + in(\theta + 2k\pi)}$

$e^{n\ln|z|} e^{in\theta_p} \underbrace{e^{in2k\pi}}$

$e^{i2m\pi} = \cos(2m\pi) + i\cancel{\sin(2m\pi)}^{0} = 1$

w is unique. there is no ambiguity.

(ii) $c = 1/n$, n: positive integer

$f(z) = z^{1/n}$ "n^{th} root" of z

$f(z) = e^{\frac{1}{n}Lnz} = e^{\frac{1}{n}[\ln|z| + i(\theta_p + 2k\pi)]} = e^{\frac{1}{n}\ln|z|} e^{i\theta_p/n} e^{i2k\pi/n}$

$e^{i\frac{2k\pi}{n}} \to k=0: 1 \quad k=1: e^{i\frac{2\pi}{n}} \neq \ldots k=n-1$: different

$\therefore$ there are n different values of $z^{1/n}$

(you can start from any k, just as long as there are n consecutive k values)

(iii) c: complex $\neq$ rational $\quad c = c_1 + ic_2$

$f(z) = z^c = e^{cLnz} = e^{(c_1 + ic_2)\{\ln|z| + i(\theta_p + 2k\pi)\}}$
$= e^{c_1\ln|z| - c_2(\theta_p + 2k\pi)} e^{i\{c_2\ln|z| + c_1(\theta_p + 2k\pi)\}}$

infinitely many values

ex Find z^{1+i}, $z = 1+i$.

$|z| = \sqrt{2}$, $\theta_p = \frac{\pi}{4}$

$z^{1+i} = e^{(1+i)Lnz} = e^{(1+i)\{\ln\sqrt{2} + i(\pi/4 + 2k\pi)\}} = e^{\ln\sqrt{2} - (\frac{\pi}{4} + 2k\pi)} e^{i(\pi/4 + 2k\pi + \ln\sqrt{2})}$

k: all integer values