

Sarkovskii's Theorem - Part 1

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May 11, 2005

The goal of this paper is to prove Sarkovskii's theorem and its converse, important results in the study of one-dimensional dynamical systems. Sarkovskii's theorem is important because its hypothesis is very easy to satisfy, and yet it provides an interesting conclusion about the periodic points of functions. Furthermore, in proving the converse, we see a number of interesting examples of one-dimensional dynamical systems that help to guide our intuition.

We first need a brief piece of preliminary notation and then we can state and prove Sarkovskii's theorem and converse.

Given the following sequence:

$$3, 5, 7, \dots, 2 \cdot 3, 2 \cdot 5, \dots, 2^2 \cdot 3, 2^2 \cdot 5, \dots, 2^n \cdot 3, 2^n \cdot 5, \dots, 2^3, 2^2, 2, 1$$

We define an ordering on the natural numbers by $a > b$ if a appears before b in the above sequence.

Sarkovskii's Theorem: Suppose $f : \mathbf{R} \rightarrow \mathbf{R}$ is continuous. Suppose f has a periodic point of prime period k . If $k > l$ in the previously defined ordering, then f also has a periodic point of prime period l .

We have three main cases to prove: f has a periodic point q where in one case q has odd prime period, in another q has a prime period of 2^m for some $m \in \mathbf{Z}^+$, and in the last q has a prime period of $p \cdot 2^m$ for p odd and $1 \leq m$. For a proof of the first two cases, see *An Introduction to Chaotic Dynamical Systems* by Robert Devaney.

In this paper, we will prove the case where f has a periodic point with prime period $p \cdot 2^m$ for p odd and $1 \leq m$ given the case for prime period p with p odd and the case for prime period 2^a .

Proof: Suppose that f has a periodic point with prime period $p \cdot 2^m$ for p odd and $1 \leq m$. Then f^{2^m} has a periodic point with prime period p , and by the odd p case of Sarkovskii's theorem it has periodic points with prime period $q \cdot 2^a$ for all odd natural numbers q and whole numbers a such that $q > p$ or $a \geq 1$. Thus f has periodic points of prime period $q \cdot 2^a$ for all odd natural numbers q and whole numbers a such that $q > p$ or $a > m$. f must thus have a periodic point of prime period 2^{m+1} , and thus by the 2^a case of Sarkovskii's theorem it also has periodic points of prime period 2^a for $0 \leq a \leq m$.

Converse of Sarkovskii's Theorem: If $k > l$ in the previously defined ordering, then there exists a continuous function $f : \mathbf{R} \rightarrow \mathbf{R}$ with a periodic point of prime period l but none periodic points of prime period k .

We are free to instead find functions $f : I \rightarrow I$, with $I = [0, 1]$ that have this property as they can easily be extended to the real numbers. I ("the interval") will continue to be defined as it is here throughout the rest of the paper.

The converse case can be reduced to four lemmas by observing the following:

If a continuous function from the interval to itself has no periodic points of prime period b then it has no periodic points of prime period $a < b$ (in the above ordering), since if it did, then Sarkovskii's theorem would imply that it had a periodic point of prime period b . Thus we only need to prove the following:

Lemma 1: A continuous function $f : I \rightarrow I$ exists such that f has a periodic point of prime period $2n + 1$ but not one of prime period $2n - 1$ for each $n \in \mathbf{Z}^+$.

Proof: Define $f : I \rightarrow I$ as follows:

$$f = \begin{cases} \frac{1}{2} + nx & x \in [0, \frac{1}{2n}], \\ 1 + \frac{1}{2n} - x & x \in [\frac{1}{2n}, \frac{1}{2}], \\ \frac{1}{2} + 1 + \frac{1}{2n} - 2x & x \in [\frac{1}{2}, \frac{n+1}{2n}], \\ 1 - x & x \in [\frac{n+1}{2n}, 1]. \end{cases}$$

f is continuous by definition, and I claim 0 is part of a $2n + 1$ cycle. I claim $f^i(0)$ has values as follows:

$$f^i(0) = \begin{cases} 0 & i = 0, \\ \frac{1}{2} - \frac{1}{2n} \lfloor \frac{i}{2} \rfloor (\in [\frac{1}{2n}, \frac{1}{2}]) & i \text{ odd} \\ \frac{1}{2} + \frac{1}{2n} \lfloor \frac{i}{2} \rfloor (\in [\frac{n+1}{2n}, 1]) & i \text{ even, } i \neq 0. \end{cases}$$

for $0 \leq i \leq 2n$ and $f^{2n+1}(0) = 0$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x . This can be seen by induction: $f(0) = \frac{1}{2}$, thus the n odd case is true for $i = 1$. Also, $f(\frac{1}{2}) = \frac{1}{2} + \frac{1}{2n}$, thus the n even case is true for $i = 2$. Now we show the inductive cases of the n even and n odd cases. Using the definition of f , i odd $\rightarrow f^{i+2}(0) = 1 - (1 + \frac{1}{2n} - f^i(0)) = f^i(0) - \frac{1}{2n} = \frac{1}{2} - \frac{1}{2n} \lfloor \frac{i+2}{2} \rfloor$. Similarly, i even $\rightarrow f^{i+2}(0) = 1 + \frac{1}{2n} - (1 - f^i(0)) = f^i(0) + \frac{1}{2n} = \frac{1}{2} + \frac{1}{2n} \lfloor \frac{i+2}{2} \rfloor$.

Now we must show that f has no cycles of prime period $2n - 1$. For this discussion, assume a is an integer, $0 \leq a < n$. Since f is piecewise linear, we can say that $f^{2a+1}([\frac{a}{2n}, \frac{a+1}{2n}]) = [\frac{1}{2}, 1]$ and $f^{2(n-a)}([\frac{1}{2} + \frac{a}{2n}, \frac{1}{2} + \frac{a+1}{2n}]) = [\frac{1}{2}, 1]$ using our previous knowledge of the orbit of 0. We can also say that $f^{2a}([\frac{1}{2}, 1]) = [\frac{1}{2} - \frac{a}{2n}, 1]$ and $f^{2a-1}([\frac{1}{2}, 1]) = [0, \frac{1}{2} + \frac{a}{2n}]$. Thus $f^{2n-1}([\frac{a}{2n}, \frac{a+1}{2n}]) = [\frac{1}{2} - \frac{n-a-1}{2n}, 1] = [\frac{a+1}{2n}, 1]$ and $f^{2n-1}([\frac{1}{2} + \frac{a}{2n}, \frac{1}{2} + \frac{a+1}{2n}]) = [0, \frac{1}{2} + \frac{a}{2n}]$. Thus f^{2n-1} has no fixed points on $[0, \frac{1}{2}] \cup [\frac{1}{2} + \frac{1}{2n}, 1]$ (since we know $f^{2n-1}(\frac{i}{2n}) \neq \frac{i}{2n}$ for all integers i , $0 \leq i \leq 2n$).

We must now consider possible fixed points for f^{2n-1} on $[\frac{1}{2}, \frac{1}{2} + \frac{1}{2n}]$. $f^{2n-1}([\frac{1}{2}, \frac{1}{2} + \frac{1}{2n}]) = I$, and thus f^{2n-1} must have at least one fixed point on $[\frac{1}{2}, \frac{1}{2} + \frac{1}{2n}]$. But, f is monotonically decreasing on $[\frac{1}{2n}, 1]$, and thus by using knowledge of the orbit of 0 (which tells us the successive images of $[\frac{1}{2}, \frac{1}{2} + \frac{1}{2n}]$ under f) we know that f^{2n-1} is monotonically decreasing on $[\frac{1}{2}, \frac{1}{2} + \frac{1}{2n}]$ and thus there is at maximum one fixed point for f^{2n-1} in this interval, which must also be a fixed point for f (f must have at least one fixed point in $[\frac{1}{2}, \frac{1}{2} + \frac{1}{2n}]$ since $f([\frac{1}{2}, \frac{1}{2} + \frac{1}{2n}]) \supset [\frac{1}{2}, \frac{1}{2} + \frac{1}{2n}]$) and thus is not a periodic point of prime period $2n - 1$. Thus f has no periodic points of prime period $2n - 1$.

Lemma 2: A continuous function $f : I \rightarrow I$ exists such that f has a periodic point of prime period $2^k(2n+1)$ but not one of prime period $2^k(2n-1)$ for each $n, k \in \mathbf{Z}^+$.

Proof: We will introduce a way of “doubling” a continuous function $f : I \rightarrow I$. Given such an f , define the doubling, $D(f)$, of f as follows:

$$F(x) = \begin{cases} \frac{2}{3} + \frac{1}{3}f(3x) & x \in [0, \frac{1}{3}] \\ (2 - 3x)(\frac{1}{3}f(1) + \frac{2}{3}) & x \in [\frac{1}{3}, \frac{2}{3}] \\ x - \frac{2}{3} & x \in [\frac{2}{3}, 1] \end{cases}$$

F is continuous and from I to I since f has these properties. An important property to note is that if $x \in [0, \frac{1}{3}]$ then $F(x) = \frac{2}{3} + \frac{1}{3}f(3x) \in [\frac{2}{3}, 1]$. Then $F^2(x) = \frac{1}{3}f(3x)$, and thus if $f(a) = b$, then $F^2(\frac{a}{3}) = \frac{b}{3}$. I claim that f has a periodic point of prime period n if and only if $D(f)$ has a periodic point of prime period $2n$:

If f has a periodic point of prime period n , $D(f)$ has a periodic point of prime period $2n$ by the properties of F we have just shown. Suppose, conversely $D(f)$ has a periodic point p of prime period $2n$ ($n \in \mathbf{Z}$). Since $F([0, \frac{1}{3}]) \subset [\frac{2}{3}, 1]$ and $F([\frac{2}{3}, 1]) = [0, \frac{1}{3}]$, and F is monotonically decreasing on $[\frac{1}{3}, \frac{2}{3}]$, we have two options: there can be at most one fixed point in $[\frac{1}{3}, \frac{2}{3}]$, or else the orbit of p alternates between points in $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. Thus we have a subsequence of the orbit of p , call this p_i (starting with $i = 0$), such that $F^2(p_i) = p_{i+1}$, $p_n = p_0$ ($F^{2n}(p_0) = p_0$, and also p_n is the first p_i to equal p_0 since p_0 has prime period $2n$ for F). Then the point $3p_0$ has an orbit $3p_i$ under f , and thus has a prime period of n .

Thus to create a continuous function $F : I \rightarrow I$ with a point of prime period $2^k(2n+1)$ but not one of prime period $2^k(2n-1)$, we need only select a function $f : I \rightarrow I$ as above with a point of prime period $2n+1$ but not one of prime period $2n-1$ and take $F = D^k(f)$.

Lemma 3: A continuous function $f : I \rightarrow I$ exists such that f has a periodic point of prime period 2^n but not one of prime period 2^{n+1} for $n \in \mathbf{Z}^+$.

Proof: First we want to construct a function with a fixed point (prime period 1) but no periodic points of prime period 2. A function that evidently has this property is $f(x) = x$, since every point is fixed. Then we can use the doubling construction from the previous lemma to construct functions with periodic points of prime period 2^n but not periodic points of prime period 2^{n+1} .

Lemma 4: A continuous function $f : I \rightarrow I$ exists such that f has a periodic point of prime period $3 \cdot 2^n$ but not one of prime period $(2m-1)2^{n-1}$ for each $n \in \mathbf{Z}^+$ and any $m \in \mathbf{Z}^+$.

Proof: First we want to construct a function with a periodic point of prime period $6 = 2 \cdot 3$ but no periodic points of odd period (greater than 1). Define $f : I \rightarrow I$ as follows:

$$f(x) = \begin{cases} \frac{1}{2} + x & x \in [0, \frac{1}{2}] \\ -2(x - \frac{1}{2}) + 1 & x \in [\frac{1}{2}, 1] \end{cases}$$

f has a periodic point of prime period 3 (the point is $x = 0$). Now consider $F = D(f)$. F has a periodic point of prime period 6. By previous logic, if F has a periodic point it must be either of even period or a fixed point. Thus F has no periodic points of odd period. To produce functions with periodic points of prime period $3 \cdot 2^n$ but none of prime periods $(2m - 1) \cdot 2^{n-1}$ we need only take $D^{n-1}(F)$.

(Note: it is not immediately evident why the last lemma must be as stated. The reason why this is the final case is that for each of the previous three lemmas there is a well defined number “immediately preceding” the number in question in the defined ordering; that is, a number a immediately precedes b if $a < b$ and $c < b \rightarrow c = a$ or $c < a$. There is no number immediately preceding $3 \cdot 2^n$ for $n \in \mathbf{Z}^+$. Thus if a continuous function f on the interval has a periodic point of prime period $3 \cdot 2^n$, in order for it to have no periodic points of prime period $a < 3 \cdot 2^n$, it is necessary that it have no periodic points of prime period $(2m - 1) \cdot 2^{n-1}$ for any $m \in \mathbf{Z}^+$, as opposed to the other three cases in which since there is an immediately preceding number, we only have to prove the function has no periodic points with a prime period equal to this immediately preceding number.)