Sarkovskii's Theorem

Genya Zaytman

May 11, 2005

1 Introduction

Sarkovskii's theorem is a remarkable result in dynamical systems, specifically in the weakness of its hypotheses. For continuous maps from the real line to itself, the theorem lets us deduce the existence of cycles of certain periods from the existence of cycles of a different period.

A special case of the theorem states that if the function has a cycle of period three it has cycles of all periods. To state the general theorem we must first define the *Sarkovskii ordering* on the natural numbers:

$$3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2 \cdot 3 \rhd 2 \cdot 5 \rhd 2 \cdot 7 \rhd \cdots \rhd 2^2 \cdot 3 \rhd 2^2 \cdot 5 \rhd 2^2 \cdot 7 \rhd \cdots$$

$$\rhd 2^3 \cdot 3 \rhd 2^3 \cdot 5 \rhd 2^3 \cdot 7 \rhd \cdots \cdots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1.$$

That is, first all odd numbers greater than one in increasing order, followed by 2 times those numbers, then by 2^2 times them, then 2^3, and so on. The lists all the natural numbers except powers of 2. Then one lists all the powers of 2 in decreasing order. Using this notation Sarkovskii's theorem states:

Theorem 1.1. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function that has a cycle of period n, than it has cycles of all periods that follow n in the Sarkovskii order.

The converse of Sarkovskii's theorem is also true.

Theorem 1.2. For each natural number n there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ that has a cycle of period n but no cycles of periods that precede n in the Sarkovskii order.

[Devaney] has a partial proof of Sarkovskii's theorem and its converse, here we finish the proof.

2 Sarkovskii's Theorem

In ([Devaney], section 1.10) Sarkovskii's theorem was shown for the cases when n is odd and when n is a power of 2. We now prove the case when $n = p2^m$.

Proof. We may assume f has no cycles of period $q2^k$ with $q > 1$ and odd and $k < m$. Now since f has a cycle of period $n = p2^m$, f^{2^m} has a cycle of period p. Therefore by the case with $n = p$ it has cycles of all odd periods greater than p and all even periods. A cycle of f^{2^m} of period $q2^\ell$ with $\ell > 0$ corresponds to a cycle of period $q2^{k+\ell}$ for f, and a cycle of odd period $q > 1$ for f^{2^m} corresponds to one for f of period $q2^k$ with $k \leq m$. By our initial assumption this period must be $q2^m$. Finally to see that we have cycles for all powers of 2 notice that the above argument has shown that we have cycles for all 2^k with $k > m$, hence applying the case with $n = 2^{m+1}$ gives us all powers of 2. □

3 Converse

Lemma 3.1. For each natural number $n > 1$, there exists a continuous function, f, with a cycle of period $2n + 1$ but not one of $2n - 1$.

Proof. Define the map $f : [0, 2n] \to [0, 2n]$ on the integers by

$$\begin{aligned} &f(0) = n \\ &f(k) = 2n + 1 - k && \text{for } 0 < k \leq n \\ &f(k) = 2n - k && \text{for } n < k \leq 2n \end{aligned}$$

so that 1 has period $2n + 1$, specifically $(0,\, n,\, n + 1,\, n - 1,\, n + 2,\, n - 2,\, n + 3,\, \cdots,\, 2n - 1,\, 1,\, 2n)$. Let f be linear between these integers.

Note that $p = n + \frac{1}{3}$ is a fixed point of f. Also observe that if $x > p$ we must have $f(x) < p$, and $1 < x < p$ implies $f(x) > p$. Hence, any cycle with odd period greater than 1 must contain a point in the interval $[0, 1]$. However, it is easy to check by induction that for $0 \leq k < n$, $f^{2k+1}([0, 1]) = [n - k, 2n]$. In particular, $f^{2n-1}([0, 1]) = [1, 2n]$. So f cannot have any cycles of period $2n - 1$. □

Lemma 3.2. Let S be the set of cycle periods of a continuous function f. Than there exists a function F the set of whose cycle periods is precisely $\{1\} \cup \{2k | k \in S\}$.

Proof. This is shown in ([Devaney], pp. 67-68). □

We are now ready to prove the converse of Sarkovskii's theorem.

Proof. First notice that if n has an immediate predecessor in the Sarkovskii ordering it suffices to show there exists a function with a cycle of period n but not one whose period precedes n in the Sarkovskii ordering.

Case 1 (n is odd and greater than 1): this is just lemma 3.1.

Case 2 ($n = m2^k$ where $m > 3$ is odd): we will show this by induction on k. For $k = 0$, this reduces to case 1. Now suppose f has a cycle of period $m2^{k-1}$ but not one of period $(m-2)2^{k-1}$. Then by lemma 3.2 there exists a function with a period-$m2^k$ cycle but not a period-$(m-2)2^k$ cycle.

Case 3 ($n = 3 \cdot 2^k$): we will also show this case by induction on k. We need a function that has a cycle of period $3 \cdot 2^k$ but no cycles of period $m2^{k-1}$ for odd $m > 1$. This is obvious if $k = 0$ since there clearly exists a function with a cycle of period 3 and cycle must have integral periods. If this is true for $3 \cdot 2^{k-1}$, this must also be true for $3 \cdot 2^k$ by lemma 3.2.

Case 4 ($n = 2^k$): we again proceed by induction on k. For $k = 0$ we must exhibit a function that has a fixed point and no other cycles, for example a constant. Now if we have a function with a cycle of period 2^{k-1} but not one of period 2^k, then by lemma 3.2 there exists a function with a cycle of period 2^k but not one of period 2^{k+1}. $\square$

References

[Devaney] R. L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Addison-Wesley. (1989).