

## 18.100A Fall 2012: Assignment 19

**Directions:** Same as for previous assignments.

**Reading: 22.3,4** Continuity of uniform limits. Integration of series term-by-term.

**Problem 1.** (2) Prove the series  $\sum_{n=1}^{\infty} \frac{e^{-nx}}{(x+n)^2}$  converges for  $x \in [0, \infty)$ , and its sum  $f(x)$  is continuous on this interval.

(Use simple comparison-type arguments, not tests meant primarily for power series. Verify hypotheses of any theorems being used.)

**Problem 2.** (2) Prove the series  $\sum_{n=0}^{\infty} e^{-nx}$  converges on  $(0, \infty)$  to a continuous function.

(The convergence is not uniform on this interval. Give carefully the argument which gets around this difficulty. A similar argument is given in section 22.3.)

**Problem 3.** (3:2,1) Let  $f(x) = \sum_1^{\infty} \frac{\sin nx}{(n-1)!}$ .

a) Prove carefully that the integral  $\int_0^{\pi} f(x) dx$  exists, obtaining its numerical value as the sum of an infinite series. (Cite theorems which justify each step; verify hypotheses.)

b) Express the value of the integral in terms of the function:  $\sinh x = \frac{e^x - e^{-x}}{2}$ .

(Get the series for  $\sinh x$  ("cinch x", the hyperbolic sine) from the known series for  $e^x$ .)

**Problem 4.** (3) Work Problem 22-3b.

Earlier in the semester, in class,  $J_0(x)$  was introduced as the function whose graph gives at a single moment in time the diametral cross-sectional shape of a vibrating bass drumhead in a marching band, shortly after the drummer hits it exactly in the middle.

From this, one can derive a differential equation that this shape must satisfy – Bessel's ODE, given in Problem 22-2 – and by finding a series solution to this ODE, get the power series representing  $J_0(x)$ .

Actually, Bessel was an astronomer, and he discovered the function in a different context, and in the strange-looking form given in P22-3b. So the purpose of this problem is to show that the two representations of  $J_0(x)$  are equivalent.

To do this, you will need the power series form of  $J_0(x)$ , as given in P22-2, and also the standard definite integrals in 22-3a. (These are used in 18.01 and 18.02 for evaluating various multiple integrals – moments of inertia of solids having an axis of symmetry, for instance.)

The series to start with is the power series for  $\cos u$ .

After  $x \sin \theta$  is substituted for  $u$ , it is no longer a power series, but the theorems in 22.3,4 are still usable, since they are phrased in terms of general functions  $u_k(x)$ .

If you mind your  $\theta$ 's,  $x$ 's, and  $u$ 's – the usual variable  $x$  used in the problem does not play the same role as the  $x$  in the book, for instance – and don't make any errors in calculation, it should all work out in the end, despite some possible dark moments in the middle.

Cite the key theorems you are using; verify their hypotheses hold here.

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