18.100B Problem Set 5

Due Friday October 20, 2006 by 3 PM

Problems:

1) Let $\mathcal{M}$ be a complete metric space, and let $X \subseteq \mathcal{M}$. Show that X is complete if and only if X is closed.

2) a) Show that a sequence in an arbitrary metric space (x_n) converges if and only if the 'even' and 'odd' subsequences (x_{2n}) and (x_{2n-1}) both converge to the same limit.
   b) Show that a sequence in an arbitrary metric space (x_n) converges if and only if the subsequences (x_{2n}), (x_{2n-1}), and (x_{5n}) all converge.

3) If (x_n) and (y_n) are two bounded sequences of real numbers, show that
   a) $\limsup (x_n + y_n) \leq \limsup x_n + \limsup y_n$
   b) $\liminf (x_n + y_n) \geq \liminf (x_n) + \liminf (y_n)$
   Moreover, show that if (x_n) converges, then both inequalities are actually equalitites.
   (*Hint:* Pick a subsequence of $(x_n + y_n)$ that converges, then, from these x_{n_k}'s pick a subsequence that converges and do the same for the y_{n_k}'s)

4) The 'sequence of averages' of a sequence of real numbers (x_n) is the sequence (a_n) defined by
$$a_n = \frac{x_1 + x_2 + \ldots + x_n}{n}$$
   If (x_n) is a bounded sequence of real numbers, then show that
$$\liminf x_n \leq \liminf a_n \leq \limsup a_n \leq \limsup x_n.$$
   In particular, if $x_n \to x$ then show that $a_n \to x$. Does the convergence of (a_n) imply the convergence of (x_n)?
   (*Hint:* Fix $\varepsilon > 0$, let $x^* = \limsup x_n$ and set $K = \{k \in \mathbb{N} : x_k \geq x^* + \varepsilon\}$. K is finite (why?), define $\mathcal{S}_n = \{i \in \mathbb{N} : i \in K \text{ and } i \leq n\}$ and $\mathcal{T}_n = \{i \in \mathbb{N} : i \notin K \text{ and } i \leq n\}$ and define the sequences (s_n), (t_n) by
$$s_n = \sum_{i \in \mathcal{S}_n} x_i, \qquad t_n = \sum_{i \in \mathcal{T}_n} x_i$$
   Explain why $a_n = \frac{s_n}{n} + \frac{t_n}{n}$, $\frac{s_n}{n} \to 0$ and $\frac{t_n}{n} \leq x^* + \varepsilon$ for any n. Then use the previous exercise to show that $\limsup a_n \leq x^* + \varepsilon$. Hence $\limsup a_n \leq x^*$ (why?))

5) Consider any sequence (x_n) defined by choosing $0 < x_1 < 1$ and then defining $x_{n+1} = 1 - \sqrt{1 - x_n}$ for $n \geq 0$. Show that x_n is a decreasing sequence converging to zero. Also, show that $\frac{x_{n+1}}{x_n} \to \frac{1}{2}$.

6) The Greeks thought that the number Φ, known as the Golden Mean, was the ratio of the sides of the most aesthetically pleasing rectangles.
   Imagine a line segment A divided into two smaller line segments B and C, with lengths a, b, and c respectively and $b > c$. If the proportion between a and b is the same as the proportion between b and c, then we call this proportion Φ.

a) Show that with a, b, and c as above, $\Phi = \frac{b}{c}$ satisfies $\Phi^2 = \Phi + 1$. Conclude that $\Phi = \frac{1+\sqrt{5}}{2}$.

b) Show that:

$$\Phi \; = \; 1 + \cfrac{1}{1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\dots}}}}$$

Hint: Define $x_1 = 1$ and $x_{n+1} = 1 + \frac{1}{x_n}$.

c) Show that:

$$\Phi \; = \; \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}$$

Hint: Define $y_1 = 1$ and $y_{n+1} = \sqrt{1 + y_n}$.

d) The Fibonacci sequence is defined by $z_1 = 1$, $z_2 = 1$, and $z_{n+2} = z_{n+1} + z_n$. Show that the sequence of ratios of succesive elements, $\frac{z_{n+1}}{z_n}$, converges to Φ.

Φ shows up a lot in nature. One reason for this might be that it is the 'most irrational number'. For more information about this, check out the links section of the course webpage.

Extra problems:

1) Prove that $\lim x_n = x$ if and only if every subsequence of (x_n) has a subsequence that converges to x.

2) If (x_n) is a sequence of strictly positive real numbers, show that

$$\liminf \frac{x_{n+1}}{x_n} \leq \liminf \sqrt[n]{x_n} \leq \limsup \sqrt[n]{x_n} \leq \limsup \frac{x_{n+1}}{x_n}$$

3) Fix a positive number α. Choose $x_1 > \sqrt{\alpha}$ and define x_n for $n > 1$ by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right)$$

Prove that (x_n) decreases monotonically and that $\lim x_n = \sqrt{\alpha}$. Show that, if $\varepsilon_n = x_n - \sqrt{\alpha}$, then

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$$

so that, setting $\beta = 2\sqrt{\alpha}$,

$$\varepsilon_{n+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n}$$