# 18.100B Problem Set 7

## Due Friday November 3, 2006 by 3 PM

**Problems:**

1) Consider an infinite series with alternating signs

$$a_0 - a_1 + a_2 - a_3 + a_4 - \ldots = \sum (-1)^n a_n.$$

Prove the Leibnitz criteria for convergence:

If $a_i > 0$ for all $i$, $a_{i+1} \leq a_i$ and $\lim_{i\to\infty} a_i = 0$, then the series converges.

Notice that this applies to the series $\sum \pm\frac{1}{n}$ which is not absolutely convergent.
(*Hint:* Show that $s_1 \leq s_3 \leq s_5 \leq \ldots \leq s_4 \leq s_2 \leq s_0$ and $|s_{n+k} - s_n| \leq a_{n+1}$.)

2) We say that a rational number $r > 0$ is written in reduced form $r = p/q$ if $p$ and $q$ are positive integers with no common factor. Consider the function defined on $(0, 1)$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q} & \text{if } x = p/q \text{ in reduced form} \end{cases}$$

Prove that $f$ is continuous at every irrational $x$, but discontinuous at every rational $x$. (Notice this shows that a function can be continuous at a point without being continuous in any neighborhood of that point.)

3) Let $f$ and $g$ be continuous mappings of a metric space $\mathcal{M}$ into a metric space $\mathcal{N}$, and let $Q \subseteq \mathcal{M}$ be a dense subset of $\mathcal{M}$.
a) Prove that $f(Q)$ is dense in $f(\mathcal{M})$.
b) Show that if $f(x) = g(x)$ for every $x \in Q$, then $f(x) = g(x)$ for every $x \in \mathcal{M}$.

4) Suppose that $f : E \to \mathbb{R}$ is a continuous map from some subset $E \subseteq \mathbb{R}$.
a) Show that it is possible to have $E$ bounded and $f(E)$ unbounded.
b) Show that if $f$ is uniformly continuous and $E$ is bounded, then $f(E)$ must be bounded.
c) Show that it is possible to have $f$ uniformly continuous, $E$ unbounded, and $f(E)$ unbounded.

5) Suppose that $f : \mathcal{M} \to \mathcal{N}$ is a uniformly continuous mapping between metric spaces.
a) Prove that if $(x_n)$ is a Cauchy sequence in $\mathcal{M}$, then $(f(x_n))$ is a Cauchy sequence in $\mathcal{N}$.
b) Use the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$ to show that it is possible for a continuous function to send Cauchy sequences to Cauchy sequences without being uniformly continuous.

6) Let $E$ be a non-empty subset of a metric space $\mathcal{M}$, define the distance from $x \in X$ to $E$ by

$$d_E(x) = \inf_{z\in E} d(x, z)$$

Prove that $d_E$ is a uniformly continuous function on $X$, by showing that

$$|d_E(x) - d_E(y)| \leq d(x, y).$$

7) Suppose $K$ and $F$ are disjoint subsets of $\mathcal{M}$, such that $F$ is closed and $K$ is compact. Prove that there exists a $\delta > 0$ such that $d(p,q) > \delta$ whenever $p \in F$ and $q \in K$. Show that the conclusion can fail if we only assumed that $K$ was closed, instead of compact.
(*Hint:* $d_F$ is a continuous positive function on $K$)

**Extra problems:**

1) Suppose $X$, $Y$, and $Z$ are metric spaces, and $Y$ is compact. Let

$$X \xrightarrow{f} Y \xrightarrow{g} X$$

and assume that $g$ is one-to-one.
a) Show that if $g$ is continuous and $g \circ f$ is continuous then $f$ is continuous.
b) Show that if $g$ is continuous and $g \circ f$ is uniformly continuous then $f$ is uniformly continuous.
c) Show that compactness of $Y$ is necessary, even if $X$ and $Z$ are compact.

2) A real-valued function $f$ defined in $(a,b)$ is said to be *convex* if

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$

for any $x, y \in (a,b)$ and $\lambda \in (0,1)$.
a) Prove that a convex function is automatically continuous.
b) Prove that every increasing convex function of a convex function is convex.
c) Assume $g$ is a continuous real function defined on $(a,b)$ and satisfying

$$g\left(\frac{x+y}{2}\right) \leq \frac{g(x)+g(y)}{2}$$

for all $x, y \in (a,b)$. Prove that $g$ is convex.

3) In class we showed that a continuous function $f : [a,b] \to \mathbb{R}$ is uniformly continuous. Prove this by either:
a) Assume it is false, so for some $\varepsilon$ no choice of $\delta$ works everywhere. Find, for each $n \in \mathbb{N}$ a point $x_n$ where $\delta = \frac{1}{n}$ does not work. Extract a convergent subsequence, $(x_{n_k})$ and derive a contradiction from the convergence of $(f(x_{n_k}))$.
b) Fix $\varepsilon > 0$, and for each $x \in [a,b]$ let $\delta(x)$ be the length of the largest open interval $I$ centered at $x$ such that $|f(z) - f(y)| \leq \varepsilon$ whenever $z, y \in I$ (really $\delta(x)$ is defined as a supremum of course). Show that $\delta(x) > 0$ and $\delta(x)$ is continuous. Because $[a,b]$ is compact, $\delta(x)$ must achieve a minimum, say $\delta_0$. Show that $\delta_0$ works in the definition of uniform continuity.