18.100B Problem Set 9

Due Friday December 1, 2006 by 3 PM

Problems:

1) Let $f_n(x) = 1/(nx+1)$ and $g_n(x) = x/(nx+1)$ for $x \in (0,1)$ and $n \in \mathbb{N}$. Prove that f_n converges pointwise but not uniformly on $(0,1)$, and that g_n converges uniformly on $(0,1)$.

2) Let $f_n(x) = x/(1+nx^2)$ if $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Find the limit function f of the sequence (f_n) and the limit function g of the sequence (f_n'). Prove that $f'(x)$ exists for every x but that $f'(0) \neq g(0)$. For what values of x is $f'(x) = g(x)$? In what subintervals of $\mathbb{R}$ does $f_n \to f$ uniformly? In what subintervals of $\mathbb{R}$ does $f_n' \to g$ uniformly?

3) Let $\mathcal{M}$ be a metric space and (f_n) a sequence of functions defined on a subset $E \subseteq \mathcal{M}$.
   We say that (f_n) is **uniformly bounded** if there exists a constant M such that $|f_n(x)| \leq M$ for every $n \in \mathbb{N}$ and $x \in E$.
   Prove that if (f_n) is a sequence of bounded real valued functions that converges uniformly to a function f, then (f_n) is uniformly bounded. Prove that in this case f is also bounded. If (f_n) is a sequence of bounded functions converging pointwise to f, need f be bounded?

4) Prove that if $f_n \to f$ uniformly and $g_n \to g$ uniformly on a set E then
   a) $f_n + g_n \to f + g$ uniformly on E.
   b) If each f_n and each g_n is bounded on E, prove that $f_n g_n \to fg$ uniformly.

5) Define two sequences (f_n) and (g_n) as follows:

$$f_n(x) = x\left(1 + \frac{1}{n}\right) \quad \text{if } x \in \mathbb{R},\ n \geq 1$$

$$g_n(x) = \begin{cases} \frac{1}{n} & \text{if } x = 0 \text{ or } x \text{ is irrational} \\ q + \frac{1}{n} & \text{if } x \in \mathbb{Q} \text{ and } x = \frac{p}{q} \text{ in reduced form} \end{cases}$$

   Show that, on any interval $[a,b]$ both f_n and g_n converge uniformly, but $f_n g_n$ does not converge uniformly (cf. problem 4b).

6) Assume that (f_n) is a uniformly bounded sequence of functions converging uniformly to f on a set E, define M as in problem 3. Let g be continuous on $[-M, M]$, prove that $g \circ f_n \to g \circ f$ uniformly on E.

7) a) Show that the sequence of polynomials defined inductively by

$$P_0(x) = 0$$

$$P_{n+1}(x) = P_n(x) + \frac{1}{2}\left(x - P_n^2(x)\right)$$

   converges uniformly on the interval $[0,1]$ to the function $f(x) = \sqrt{x}$.
   b) Deduce that there exists a sequence of polynomials converging uniformly on $[-1,1]$ to the function $f(x) = |x|$.

Extra problems:

Some everywhere continuous, nowhere differentiable functions.

1) (John McCarthy) Consider the function $g : \mathbb{R} \to \mathbb{R}$ satisfying $g(x) = g(x+4)$ for ever x, and

$$g(x) = \begin{cases} 1+x & \text{for } -2 \le x \le 0 \\ 1-x & \text{for } 0 \le x \le 2 \end{cases}$$

and define

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} g\left(2^{2^n} x\right)$$

Show that f is continuous. Show that f is nowhere differentiable as follows: Take $\Delta x = \pm 2^{-2^k}$, choosing whichever sign makes x and $x + \Delta x$ be on the same linear segment of $g\left(2^{2^k} x\right)$. Show that

a) $\Delta\left(2^{2^n} x\right) = 0$ for $n > k$, since $g\left(2^{2^n} x\right)$ has period $4 \cdot 2^{-2^n}$

b) $|\Delta g\left(2^{2^k} x\right)| = 1$

c) $|\Delta \sum_{n=1}^{k-1} 2^{-n} g\left(2^{2^n} x\right)| \le (k-1) \max |\Delta g\left(2^{2^n} x\right)| \le (k-1)\, 2^{2^{k-1}} 2^{-2^k} < 2^k 2^{-2^{k-1}}$

Conclude that $|\Delta f / \Delta x| \ge 2^{-k} 2^{2^k} - 2^k 2^{2^{k-1}}$ which goes to infinity with k, and hence f is nowhere differentiable.

2) (Van der Waerden following Billingsley) Let $a_0(x)$ denote the distance from x to the nearest integer, $a_k(x) = 2^{-k} a_0\left(2^k x\right)$, and define

$$f(x) = \sum a_k(x).$$

a) Prove that f is everywhere continuous.

b) Prove that if a function h has a derivative at x and $u_n \le x \le v_n$ are such that $u_n < v_n$ and $u_n - v_n \to 0$ then

$$\frac{h(v_n) - h(u_n)}{v_n - u_n} \to h'(x)$$

c) Prove that f is nowhere differentiable as follows: Notice that if u is a dyadic number of order n (i.e., of the form $\frac{i}{2^n}$ for some integer i) then $2^k u$ is an integer for $k \ge n$ and

$$f(u) = \sum_{k=0}^{n-1} a_k(u).$$

Fix x and let u_n, v_n be succesive dyadics of order n (i.e., $v_n - u_n = 2^{-n}$) such that $u_n \le x < v_n$. Show that

$$\frac{f(v_n) - f(u_n)}{v_n - u_n} = \sum_{k=0}^{n-1} \frac{a_k(v_n) - a_k(u_n)}{v_n - u_n}$$

Show that each term on the right hand side is either a 1 or a -1 and conclude that the left hand side can not converge to a finite limit.

For more examples, see the Related Resources section of the course.